The Classification Theorem for Compact Surfaces

by

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Abstract

The Classification of Surfaces is one of the problems which gave rise to the modern topology. It has become one of the signature theorems of the area, which now is called algebraic topology. It states that any closed connected surface is homeomorphic to the sphere, the connected sum of tori, or the connected sum of projective planes. In this thesis we are going to go over the geometric, topological, and algebraic tools necessary for understanding, proving and using the theorem together with some useful examples of surfaces.

Thesis itself consists of three chapters. The first part talks about homotopy theory and defines the fundamental group, which is an algebraic invariant between topological spaces. In addition, we learn some basic ways of calculating the fundamental group for some easy-to-imagine examples. The second chapter introduces free groups and free products, which altogether let us calculate the fundamental group in more complex cases. The third and final chapter introduces the geometric ideas behind the classification theorem, which includes polygonal regions and labelling schemes together with operations on them. As a result, we overview the construction of any two-dimensional compact surface and classification theorem as a main goal.

Acknowledgements

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Chapter 1

The Fundamental Group

1.1 Homotopy of Paths

Two continuous functions from one topological space to another are called *homotopic* if one can be "continuously deformed" into the other. Such a deformation is called a homotopy between the two functions and that is what this chapter is about.

Definition 1.1.1. Continuous maps $f_0, f_1 : X \to Y$ are said to be **homotopic**, which is denoted by $f_0 \simeq f_1$, when there is a continuous map $F : X \times I \to Y$, called a **homotopy** from f_0 to f_1, such that for all points $x \in X$ we have $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. With this, we can define $f_{t_0}(s) = F(s, t_0)$ to be a path from x_0 to x_1 obtained at $t = t_0$.
If f is homotopic to a constant map, i.e., if $f \simeq const_y$ for some $y \in Y$, then we say that f is **nulhomotopic**.

Theorem 1.1.2. *The homotopy relation $\simeq$ is an equivalence relation on the set $Map(X, Y)$ of continuous functions from X to Y.*

Proof. Let $f, f_0, f_1, f_2 : X \to Y$ be continuous maps. We need to check all the conditions of an equivalence relation:

- Reflexivity ($f \simeq f$):
  Take the map $F : X \times I \to Y$, $F(x, t) = f(x)$. Since for all $t \in I$ and for all $x \in X$ we have $F(x, t) = f(x)$, F is a homotopy from f to f.

- Symmetry($f_0 \simeq f_1 \Rightarrow f_1 \simeq f_0$):
  Let $F : X \times I \to Y$ be a homotopy from f_0 to f_1, i.e. $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Take $G : X \times I \to Y$ such that $G(x, t) = F(x, 1 - t)$. This function is continuous since it is defined as a composition of continuous functions. Moreover, $G(x, 0) = F(x, 1) = f_1(x)$ and $G(x, q) = F(x, 0) = f_0(x)$. Thus, G is a homotopy from f_1 to f_0.

- Transitivity($f_0 \simeq f_1$ and $f_1 \simeq f_2 \Rightarrow f_0 \simeq f_2$):
  Let $F_1 : X \times I \to Y$ such that $F_1(x, 0) = f_0(x)$ and $F_1(x, 1) = f_1(x)$ be a homotopy from

f_0 to f_1. Also, let $F_2 : X \times I \to Y$ such that $F_2(x, 0) = f_1(x)$ and $F_2(x, 1) = f_2(x)$ be a homotopy from f_1 to f_2. Take

$$F_{12}(x,t) = \begin{cases} F_1(x, 2t) & 0 \leqslant t \leqslant 1/2, \\ F_2(x, 2t-1) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

By pasting lemma A.0.28, its components are continuous and since at $t = 1/2$ we have $F_1(x, 1) = f_1(x)$ and $F_2(x, 1-1) = F_2(x, 0) = f_1(x)$, F_{12} is well-defined and a homotopy from f_0 to f_2.

□

We shall denote the homotopy class of a continuous map $f : X \to Y$ by $[f]$. That is, $[f] = \{g \in Map(X,Y) \mid g \simeq f\}$. Moreover, we shall denote the set of homotopy classes of continuous maps from X to Y by $[X, Y] = Map(X, Y)/\simeq$.

Example 1.1.3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be any two continuous, real functions. To show that $f \simeq g$, consider a function $F(x, t) = (1-t) \cdot f(x) + t \cdot g(x)$. Being a composition of continuous functions F is continuous. Moreover, $F(x, 0) = (1-0) \cdot f(x) + 0 \cdot g(x) = f(x)$ and $F(x, 1) = (1-1) \cdot f(x) + 1 \cdot g(x) = g(x)$. Thus, F is a homotopy between f and g. In particular, this example shows that any continuous map $f : \mathbb{R} \to \mathbb{R}$ is nulhomotopic.

Let's consider the special case in which f is a path in X. Recall that if $f : [0, 1] \to X$ is a continuous map such that $f(0) = x_0$ and $f(1) = x_1$, we say that f is a **path** in X from x_0 from x_1. We call x_0 the **initial point** and x_1 the **final point** of the path f.

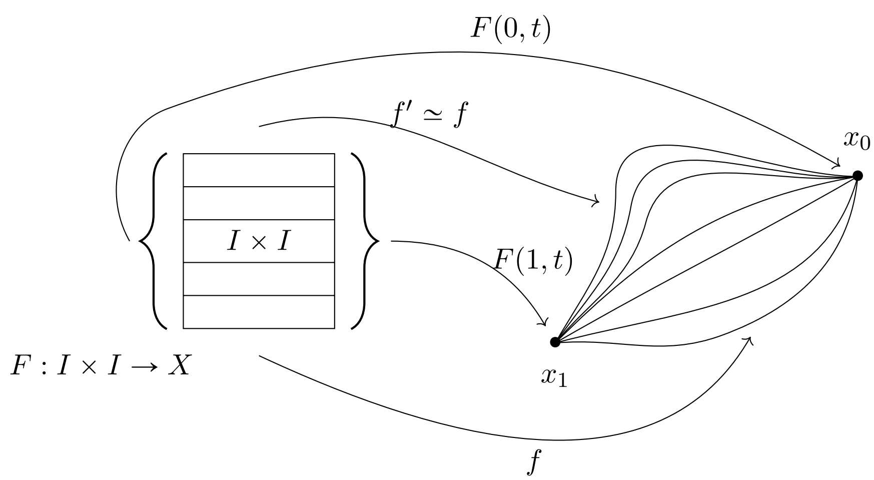


Figure 1.0.0: the path f can be continuously deformed into f' by a continuous deformation F.

Definition 1.1.4. Two paths f_0 and f_1 from $I = [0,1]$ to X are **path homotopic** if they have the same initial point x_0 and the final point x_1, and if there is a continuous map $F : I \times I \to X$ such that $\forall s \in I$ and $\forall t \in I$:

$$F(s,0) = f_0(s), \qquad F(0,t) = x_0,$$
$$F(s,1) = f_1(s), \qquad F(1,t) = x_1.$$

F in this case is called a **path homotopy** between f_0 and f_1. If f_0 is path homotopic to f_1, we write $f_0 \simeq_p f_1$. See Figure 1.0.0.

Since the path homotopy is a special case of homotopy, one can conduce the following:

Theorem 1.1.5. *The path-homotopy relation $\simeq_p$ is an equivalence relation.*

Example 1.1.6. Let f_0 and f_1 be any two maps of a space X into $\mathbb{R}^2$. Take $F(x,t) = (1-t)f_0(x) + tf_1(x)$ as in Example 1.1.3. See Figure 1.1.0. We already know F is a homotopy map between f_0 and f_1. This specific description of a homotopy is called a **straight-line (linear) homotopy** since for any $p \in X$ it moves the point $f_0(p)$ to the point $f_1(p)$ along the straight line segment. If f_0 and f_1 are both paths, then F will be a path-homotopy from f_0 to f_1.

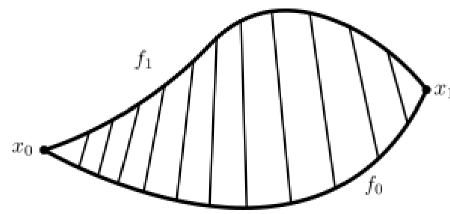


Figure 1.1.0: Straight-line homotopy between f_0 and f_1.

More generally, let A be any convex subset of $\mathbb{R}^2$. Recall that a set C is convex if the line segment between any two points in C lies in C, i.e., $\forall x_1, x_2 \in C$, $\forall \theta \in [0,1]$, $\theta x_1 + (1-\theta)x_2 \in C$. Then any two paths f_0 and f_1 between x_0 and x_1 in A are path-homotopic, since for all $s, t \in A$, by definition of convex set, we have $F(s,t) \in A$ and

$$F(s,0) = f_0(s), \qquad F(0,t) = (1-t)f_0(0) + tf_1(0) = f_0(0),$$
$$F(s,1) = f_1(s), \qquad F(1,t) = (1-t)f_0(1) + tf_1(1) = f_0(1).$$

Definition 1.1.7. Given paths f and g such that $f(1) = g(0)$, the **product path** $f * g$ (also called **composition** or **concatenation**) is given by

$$f * g(s) = \begin{cases} f(2s) & 0 \leqslant s \leqslant 1/2, \\ g(2s-1) & 1/2 \leqslant s \leqslant 1. \end{cases}$$

Intuitively, the composition law is just given by following one path, and then the other with twice the speed, as shown in Figure 1.1.1.

Because we need the endpoint of one path to be the beginning point for the other for composition, the set of paths does not form a group. However, if we assume that the paths start and end at the same point (loops), then they do. We will take a closer look at this group in the next section!

The operation $*$ can be applied to homotopy classes as well. Once again, let $f : I \to X$ be a path from x_0 to x_1 and let $g : I \to X$ be a path from x_1 to x_2. Define $[f] * [g] := [f * g]$.

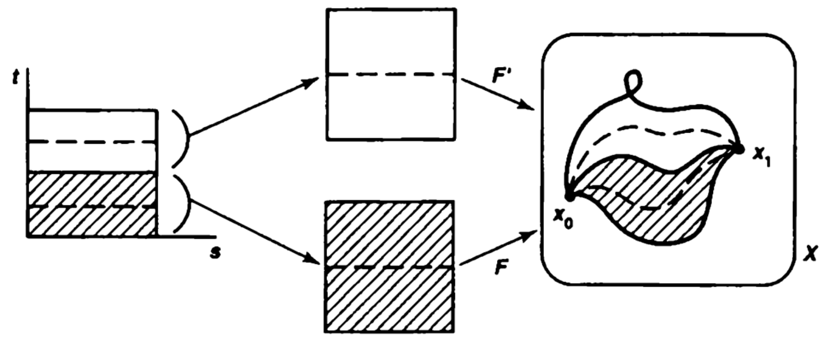


Figure 1.1.1: Construction of a concatenation [Mun00].

Theorem 1.1.8. *The operation $*$ between homotopy classes is well-defined.*

Proof. Let $f' \in [f]$ and $g' \in [g]$. Because $[f'] = [f]$ and $[g'] = [g]$, we need to check if $[f'] * [g'] = [f] * [g]$. Since f and f' are path-homotopic, there exists a path homotopy F from f to f'. Likewise, there exists a path homotopy G from g to g'. Define $H : I \times I \to X$ such that:

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leqslant s \leqslant 1/2, \\ G(2s-1,t) & 1/2 \leqslant s \leqslant 1. \end{cases}$$

Now we show that H is a path homotopy between $f * g$ and $f' * g'$, which are both paths from x_0 to x_2. We know H is continuous by the pasting lemma A.0.28. Let's check the conditions from definition of the path-homotopy:

$$H(s,0) = \begin{cases} F(2s,0) & 0 \leqslant s \leqslant 1/2, \\ G(2s-1,0) & 1/2 \leqslant s \leqslant 1, \end{cases} \qquad H(0,t) = F(0,t) = x_0,$$

$$H(s,1) = \begin{cases} F(2s,1) & 0 \leqslant s \leqslant 1/2, \\ F(2s-1,1) & 1/2 \leqslant s \leqslant 1, \end{cases} \qquad H(1,t) = G(1,t) = x_2.$$

The function H is therefore a path homotopy between $f * g$ and $f' * g'$. Thus, $[f * g]$ is independent of the class representatives f and g and the operation $*$ is well-defined on equivalence classes. $\square$

Lemma 1.1.9. Let $k : X \to Y$ be a continuous function (map) and F be a path homotopy in X between paths f and f'. Then $k \circ F$ is a path homotopy in Y between paths $k \circ f$ and $k \circ f'$.

Proof. The function $k \circ F$ is continuous by being a composition of continuous functions. Let x_0 and x_1 be respectively the initial and the final point of both f and f'. Then from the definition, we get the following:

$$\begin{aligned} k \circ F(s,0) &= k(F(s,0)) = k(f(s)) = k \circ f(s), \\ k \circ F(s,1) &= k(F(s,1)) = k(f'(s)) = k \circ f'(s), \\ k \circ F(1,t) &= k(F(1,t)) = k(x_1), \\ k \circ F(0,t) &= k(F(0,t)) = k(x_0), \end{aligned}$$

which means that $k \circ F$ is indeed the required homotopy. □

Lemma 1.1.10. Let f and g be paths such that $f(1) = g(0)$ and $k : X \to Y$ be a map. Then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

Proof. Let's start with the right side of the equality and use Definition 1.1.7:

$$(k \circ f) * (k \circ g)(s,t) = \begin{cases} k \circ f(2s,t) & 0 \leqslant s \leqslant 1/2, \\ k \circ g(2s-1,t) & 1/2 \leqslant s \leqslant 1. \end{cases}$$

which is equal to $\big(k \circ (f * g)\big)(s,t)$. □

Theorem 1.1.11. *Let $f : I \to X$ be a path from x_0 to x_1, $g : I \to X$ be a path from x_1 to x_2, and $h : I \to X$ be a path from x_2 to x_3. Define $\bar{f} : I \to X$ such that $\bar{f}(s) = f(1-s)$. Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all the points of I to one point x of X. The operation $*$ between homotopy classes has the following properties:*

- *Identity Elements:* $[e_{x_0}] * [f] = [f]$ *and* $[f] * [e_{x_1}] = [f]$.
- *Inverses:* $[f] * [\bar{f}] = [e_{x_0}]$ *and* $[\bar{f}] * [f] = [e_{x_1}]$.
- *Associativity:* $([f]) * [g]) * [h] = [f] * ([g] * [h])$.

Proof. Let's prove separately all the parts of the theorem:
<u>Identity elements</u>: We want to show $(e_{x_0} * f) \simeq_p f$. Let $e_0 : I \to I$ be the constant path at 0 and let $i : I \to I$ be the identity path. Because I is convex and both paths $e_0 * i$ and i are paths from 0 to 1 by Example 1.1.6 these two paths are path-homotopic with homotopy F. By Lemma 1.1.10, we know $f \circ (e_0 * i) = (f \circ e_0) * (f \circ i) = e_{x_0} * f$. Then $F \circ f$ is a homotopy between $f \circ i$ and $e_{x_0} * f$. Similar argument can be done to show the argument towards the right identity. It follows similarly $[f] * [e_{x_1}] = [f]$.
<u>Inverses</u>: We want to show $(f * \bar{f}) \simeq_p e_{x_0}$. Given a path f in X from x_0 to x_1, let $\bar{f}$ be a path from x_1 to x_0 such that $\bar{f}(s) = f(1-s)$. We know $f * \bar{f} = (f \circ i) * (f \circ \bar{i}) = f \circ (i * \bar{i})$. Similarly, we know $e_{x_0} = f * e_0$. Since $i * \bar{i}$ and e_0 are loops in I based at 0 and I is convex, there exists the straight line path homotopy F between $i * \bar{i}$ and e_0. Then by Lemma 1.1.9, $f \circ F$ is a path homotopy between $f \circ (i * \bar{i}) = f * \bar{f}$ and $f * e_0 = e_{x_0}$. Therefore, as before, we find $[f] * [\bar{f}] = [e_{x_0}]$. By similar reasoning, $[\bar{f}] * [f] = [e_{x_1}]$.

Associativity: By the definition of the concatenation one can write

$$(f * g) * h = \begin{cases} f(4s) & 0 \leqslant s \leqslant 1/4, \\ g(4s) & 1/4 \leqslant s \leqslant 1/2, \\ h(2s) & 1/2 \leqslant s \leqslant 1, \end{cases} \qquad f * (g * h) = \begin{cases} f(2s) & 0 \leqslant s \leqslant 1/2, \\ g(4s) & 1/2 \leqslant s \leqslant 3/4, \\ h(4s) & 3/4 \leqslant s \leqslant 1. \end{cases}$$

Let $k : I \to I$ be a map: $k(s) = \begin{cases} s/2 & 0 \leqslant s \leqslant 1/2, \\ s - 1/4 & 1/2 \leqslant s \leqslant 3/4, \\ 2s - 1 & 3/4 \leqslant s \leqslant 1. \end{cases}$

Then $((f * g) * h) \circ k = f * (g * h) = (f * (g * h)) \circ i$. Because k and i are both paths from 0 to 1 in the convex space I, k and i are both path-homotopic by Example 1.1.6. Let F be a path homotopy from k to i spoken of earlier. Then, $((f * g) * h) \circ F$ is a path homotopy from $((f * g) * h) \circ k = f * (g * h)$ to $((f * g) * h) \circ i = (f * g) * h$. Thus, $f * (g * h) \simeq_p (f * g) * h$ and $[f] * ([g] * [h]) = ([f] * [g]) * [h]$. $\square$

Now that we know $*$ is associative, we know $[f_1] * [f_2] * ... * [f_n]$ is well-defined. In other words, no matter how you chop the path, you have the product of all the pieces will give you the same result. And one can use a very smart chopping in some cases!

1.2 The Fundamental Group

Definition 1.2.1. A path $f : [0, 1] \to X$ is called a **loop** if $f(0) = f(1)$. It is said to be **based at x** if $f(0) = f(1) = x$. Moreover, a loop is **nulhomotopic** if it is homotopic to the constant loop, i.e., the loop $f : I \to X$ given by $f(t) = x_0$ for all t.

With the notion of loops, we can now talk about the group defined under the concatenation:

Definition 1.2.2. The fundamental group or the **first homotopy group of** X, $\pi_1(X; x_0)$, is the set of equivalence classes of loops $f : I \to X$ based at x_0.

Theorem 1.2.3. *The fundamental group is a group under composition of loops.*

Proof. Certainly composition is an operation taking loops to loops. We first look to see that composition is well defined on homotopy classes. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then by composing the homotopies we get a homotopy of $f_0 * g_0$ to $f_1 * g_1$. All other properties of a group come from Theorem 1.1.11. $\square$

Example 1.2.4. The space $\pi_1(\mathbb{R}^n, x_0)$, where $x_0 \in \mathbb{R}^n$, has trivial fundamental group. To see this, we have to show every loop is homotopic to the constant loop. For a loop $f : I \to \mathbb{R}^n$ at x_0, consider the straight-line homotopy $F(s, t) = t \cdot f(s) + (1 - t) \cdot x_0$. It defines a homotopy between f and the trivial loop.

In particular, the unit ball B^n in $\mathbb{R}^n$: $B^n = \{\mathbf{x} | x_1^2 + x_2^2 + ... + x_n^2 \leqslant 1\}$ has trivial fundamental group since all loops at x_o in the ball are nulhomotopic.

Definition 1.2.5. Let α be a path in X from x_0 to x_1. Define a **"α-hat"** map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ such that $\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha]$.

This function is well-defined from Theorem 1.1.8. Then if f is a loop based at x_0 then $\hat{\alpha}([f])$ is a loop based at x_1. In other words, we now have a way of "moving" from one points to another using the path between them, like shown on Figure 1.2.1.

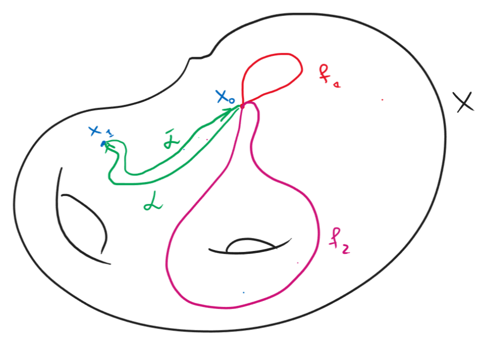


Figure 1.2.1: Both loops f_1 and f_2 which are based at x_0 can be transformed to be starting at x_1.

Theorem 1.2.6. *The map $\hat{\alpha}$ is a group isomorphism.*

Proof. First, note that $\hat{\alpha}$ is a homomorphism since

$$\begin{aligned} \hat{\alpha}[f] * \hat{\alpha}[g] &= ([\hat{\alpha}] * [f] * [\alpha]) * ([\overline{\alpha}] * [g] * [\alpha]) \\ &= [\hat{\alpha}] * [f] * [g] * [\overline{\alpha}] = \hat{\alpha}([f] * [g]). \end{aligned}$$

We need to show that $\hat{\alpha}$ is an isomorphism as well. Let $[h]$ and $[f]$ be elements of $\pi_1(X, x_1)$ and $\pi_1(X, x_0)$, respectively. Then

$$\hat{\overline{\alpha}}([h]) = [\overline{\overline{\alpha}}] * [h] * [\overline{\alpha}] = [\alpha] * [h] * [\overline{\alpha}] \text{ and } \hat{\alpha}\big(\hat{\overline{\alpha}})([h])\big) = [\overline{\alpha}] * ([\alpha] * [h] * [\overline{\alpha}]) * [\alpha] = [h].$$

Then, as a result, we get:

$$\hat{\overline{\alpha}}\big(\hat{\alpha}[f]\big) = [\alpha] * [\hat{\alpha}[f]] * [\overline{\alpha}] = [\alpha] * ([\overline{\alpha}] * [f] * [\alpha]) * [\overline{\alpha}] = [f].$$

as needed. □

Definition 1.2.7. A space X is **path connected** if there exists a path joining any two points (i.e., for all $x, y \in X$ there is some path $f : I \to X$ with $f(0) = x$, $f(1) = y$).

The fundamental group of a path connected space does not depend on the choice of base point.

Theorem 1.2.8. *Let X be a path connected space with $x, y \in X$. Then, we have an isomorphism of groups $\pi_1(X, x) \cong \pi_1(X, y)$.*

Proof. We can construct the isomorphism $\pi_1(X, x) \cong \pi_1(X, y)$ as follows. Start by choosing a path f_0 from x to y, i.e., $f_0 : I \to X$ with $f_0(0) = x$, $f_0(1) = y$. Then, send a loop f_1 based at x to the loop $\hat{\alpha}$, which is a loop based at y. □

Because of the theorem above, it is not particularly important to keep track of the base point if one is working with a path-connected space. For this reason, base point is usually omitted in the definition of a fundamental group of a space and we just write $\pi_1(X)$.
Note that if a space is not path connected, then for x_0 in a component of X, $\pi_1(X, x_0)$ provides no information about the other components of X. This is the reason for the study of fundamental groups being usually restricted to path connected spaces.

Definition 1.2.9. A space is **simply-connected** if it is path connected and $\pi_1(X, x) = 0$ for all points $x \in X$, i.e., every path between two points can be continuously transformed into any other such path while preserving the two endpoints.

A simply-connected space is a path connected space that has no “holes” that pass through the entire space. Such a hole would prevent some loops from being shrunk continuously into a single point.

Theorem 1.2.10. *In a simply-connected space X, any two paths that have the same initial point x_0 and endpoint x_1 are path homotopic.*

Proof. Let α and β be two paths from x_0 to x_1. Then $\alpha * \overline{\beta}$ is defined and is a loop based at x_0. Since X is a simply-connected space, this loop is path-homotopic to the constant loop e_{x_0} at x_0. So $[\alpha] = [\alpha * \overline{\beta}] * [\beta] = [e_{x_0}] * [\beta] = [\beta]$. □

We now develop methods to show that the fundamental group is a topological invariant or, in other words, a property shared by homeomorphic spaces.

Definition 1.2.11. Let $h : X \to Y$ be a continuous map between spaces X and Y with $y_0 = h(x_0)$. Then for a loop f in X based at x_0, $h \circ f : I \to Y$ is a loop in Y based at y_0. We denote this by $h : (X, x_0) \to (Y, y_0)$. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*[f] = [h \circ f]$. Then h_* is the **homomorphism induced by** h relative to the base point x_0. In the event that we consider the homomorphism induced by h relative to different base points, we denote h_* as $(h_{x_0})_*$ or $(h_{x_1})_*$, etc.

We need to show that in the definition above the map h_* is well-defined and indeed a homomorphism. The first condition is true since for $f, f' \in [f]$, there is a path homotopy F between f and f'. Then $h \circ F$ is a path homotopy between $h \circ f$ and $h \circ f'$ by Lemma 1.1.9. Moreover, since $(h \circ f) * (h \circ g) = h \circ (f * g)$, h is, in fact, a group homomorphism. The induced homomorphism has two crucial properties.

Theorem 1.2.12. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Proof. Since $(k \circ h)_*([f]) = [(k \circ h) \circ f]$, we get $(k_* \circ h_*)([f]) = k_*(h_*([f])) = k_*(h \circ f) = [k \circ (h \circ f)]$. Similarly, $i_*([f]) = [i \circ f] = [f]$. □

Theorem 1.2.13. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.*

Proof. Since h is a homeomorphism, it has an inverse $k : (Y, y_0) \to (X, x_0)$. Applying Theorem 1.2.12 we get $k_* \circ h_* = (k \circ h)_* = i_*$, where i is an identity map of (X, x_0). The same way, $h_* \circ k_* = (h \circ k)_* = j_*$, where j is an identity map of (Y, y_0). Since both i and j are the identity homomorphisms of the groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, respectively, k_* is an inverse of h_*. □

Theorem 1.2.14. *Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps. If h and k are homotopic, and the image of the base point $x_0 \in X$ remains fixed at $y_0 \in Y$ when acted upon by the homotopy, then the homomorphisms h_* and k_* are equal.*

Proof. Let $H : X \times I \to Y$ be the homotopy between h and k such that $H(x_0, t) = y_0$ for all $t \in I$. Then, by definition, $H(x, 0) = h(x)$ and $H(x, 1) = k(x)$. Consider a loop $f : I \to X$ based at x_0 and the compositions $h \circ f$, $k \circ f$ and $H \circ (f \times \mathbb{1}_I) : I \times I \to Y$:

$$I \times I \xrightarrow{f \times \mathbb{1}_I} X \times I \xrightarrow{H} Y$$

$$H(f(x), 0) = h \circ f(x) \qquad\qquad H(f(x), 1) = k \circ f(x)$$

$$H(f(0), t) = H(f(1), t) = y_0, \forall t \in I$$

Then $H \circ (f \times \mathbb{1}_I) : I \times I \to Y$ is a homotopy between $h \circ f$ and $k \circ f$. Moreover, $h_*([f]) = [h \circ f] = [k \circ f] = k_*([f])$, and, thus, $k_* = h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. □

Definition 1.2.15. A space X is **contractible** if there is a homotopy between the identity map $X \to X$ and a constant map.

Example 1.2.16. Let's look at a few facts about contractible spaces:

1. I, $\mathbb{R}^n$ and the disk D^n are contractible. For the first two spaces, define the homotopy by $F(x, t) = tx$. Then $f(x, 0) = 0$ and $f(x, 1) = x$, so F is a homotopy from the constant map 0 to the identity. To see that the disk is contractible it is enough to consider a straight-line homotopy from the points of the disk to the origin.

2. A contractible space is also path-connected. Let $F : X \times I \to X$ be a homotopy from a constant map $C_{x_0} : X \to X$ to the identity, i.e. $F(x, 0) = x_0$ and $F(x, 1) = x$ for all $x \in X$. For each point $x_1 \in X$, the function $g : I \to X$ such that $g(t) = F(x_1, t)$ gives a path from x_1 to x_0. Thus, all points of X are in the same path components as x_0, so X itself is path-connected.

3. If Y is contractible, then for any X, the set $[X, Y]$ has a single element. Let $F : Y \times I \to Y$ be a homotopy from a constant map to the identity, i.e., $F(y, 0) = y_0$ and $F(y, 1) = y$ for all $y \in X$. Then any map $g : X \to Y$ is homotopic to the constant map $g'(x) = y_0$ with a homotopy $G : X \times I \to Y$ defined by $G(x, t) = F(g(x), t)$. One can check that $G(x, 0) = y_0$ and $G(x, 1) = F(g(x), 1) = g(x)$ as needed.

4. If X is contractible and Y is path connected then $[X, Y]$ has a single element. Define F as a homotopy from part 2 of this example. For any function $g : X \to Y$, the function $f \circ F$ is a homotopy between g and a constant map $g'(x) = g(x_0)$. If Y is path connected, then any two constant maps are homotopic and, thus, any two maps from X to Y are homotopic.

The fundamental group is a covariant functor from the category Top_* of pointed topological spaces and pointed continuous maps to the category Groups of groups and group homomorphisms. For definitions from category theory, the reader is referred to the Appendix B.

1.3 Covering Spaces and the Fundamental Group of a Circle

To explore the fundamental groups of spaces more complex than $\mathbb{R}^n$, consider the following definition.

Definition 1.3.1. Let $p : E \to B$ be a continuous onto map. The open set U of B is **evenly covered by** p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets $V_\alpha \subseteq E$ such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ is a partition of $p^{-1}(U)$ into **slices** or **fibers**.

Definition 1.3.2. Let $p : E \to B$ be continuous and onto. If every point $b \in B$ has a neighborhood U, also called **trivialized neighbourhood**, that is evenly covered by p, then p is a **covering map** and E is said to be a **covering space** of B which is also called the **base space**.

A covering map over X is a map that locally looks like the projection map for some discrete space as seen on Figure 1.3.1.

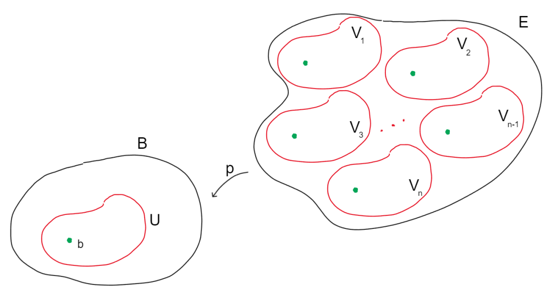


Figure 1.3.1

Example 1.3.3. The identity map $X \to X$ is always a covering map of X. In this case, we can take the entire space X to be the neighbourhood U from the definition. More generally, if F is a discrete space, then the projection $X \times F \to X$ is a covering space of X. We will call the map $X \times F \to X$ a **trivial cover**. Every covering space looks locally like a trivial cover.

Note that if $p : E \to B$ is a covering map, then for all $b \in B$ the subspace $p^{-1}(b)$ of E has the discrete topology. One can see that since each "slice" V_α, which is open in E, intersects the set $p^{-1}(b)$ in a single point, and so this point must be open in $p^{-1}(b)$.

Theorem 1.3.4. *Let $p : E \to B$ be a covering map. Then p is an open map.*

Proof. Let U be an open set in E. If $U = \varnothing$, then $p(U) = p(\varnothing) = \varnothing$ which is always open. Therefore, assume that $U \neq \varnothing$. Let $x \in p(U)$. We want to show that x is an interior point of U. Let V be a neighbourhood of x and let V_0 be a path component of $p^{-1}(V)$ or, in other words, a slice containing $p^{-1}(x)$. Then p restricted to V_0 is a homeomorphism onto V. Since V_0 is a path-connected component, it is open in E, and since U is open in, $V_0 \cap U$ is open in V_0. Since p is a homeomorphism from V_0 onto V we have that $p(V_0 \cap U)$ is open in V and is also open in B. But also $x \in p(V_0 \cap U) \subseteq p(U)$. So $x \in Int(p(U))$ and, thus, $p(U)$ is open. $\square$

Theorem 1.3.5. *The map $p : \mathbb{R} \to S^1$ given by the equation $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map.*

Proof. One can imagine the real line $\mathbb{R}$ getting wrapped around the circle with the length of a circle being 1 as shown on Figure 1.3.2. Then each interval $[n, n+1]$ makes exactly one loop around the circle. Note that p is periodic, so it is enough to discuss in detail only values of x that lie in (or near) the unit interval. Consider $x \in S^1$, and let $x_0 \in \mathbb{R}$ be any point such that $f(x_0) = x$, i.e., $f^{-1}(x) = \{x_0 + k | k \in \mathbb{Z}\}$. Let $U \subset S^1$ be a small open arc of S^1 such that $x \in U$. Then the preimage $f^{-1}(U)$ consists of a disjoint union of small intervals surrounding the points $x_0 + k$ for $k \in \mathbb{Z}$. Then U is a trivialized neighbourhood of x.

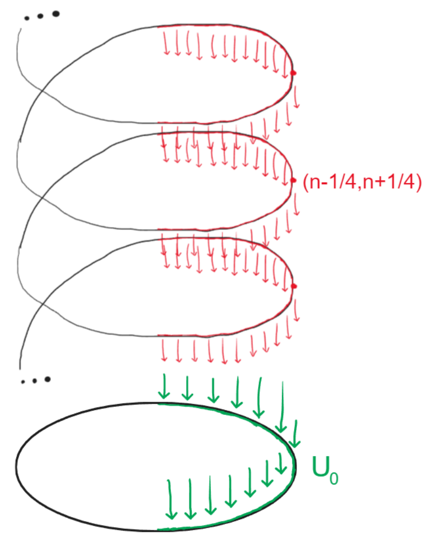


Figure 1.3.2: Visualization of the covering map of S^1.

For construction we use four open sets, U_0, U_1, U_2 and U_3, described in terms of $\mathbb{R}^2$. Take $U_0 = \{(x, y) \in S^1 : x > 0\}$. Since $\cos 2\pi x > 0$ in U_0 means that $-\pi/2 < 2\pi x < \pi/2$, every interval $(n-1/4, n+1/4)$ is getting mapped to U_0 by p. To show that every such interval is homeomorphic to U_0, note that $\sin 2\pi x$ is a monotonically increasing continuous function on any of the taken intervals as x is increasing. Therefore, we can provide an inverse continuous function $p^{-1} : S^1 \to \mathbb{R}$ where $p^{-1}(x, y) = n + \frac{1}{2\pi} \arcsin y$, which shows that p is homeomorphism on every such interval U_0. Since for all $n \in \mathbb{N}$ intervals $(n - 1/4, n + 1/4)$ are disjoint, U_0 is evenly covered by p. The same way it can be shown that $U_1 = \{(x, y) \in S^1 : y > 0\}$, $U_2 = \{(x, y) \in S^1 : x < 0\}$ and $U_3 = \{(x, y) \in S^1 : y < 0\}$ are all evenly covered by the intervals $(n, n+1/2)$, $(n+1/4, n+3/4)$ and $(n + 1/2, n + 1)$, respectively. Since all of U_i cover S^1 and each of them is evenly covered by p, p is a covering map. □

If $p : E \to B$ is a covering map, then p is a **local homeomorphism** of E with B according to Definition A.0.10. However, the condition that p is a local homeomorphism is not enough to claim that p is covering map.

Example 1.3.6. The map $p|_{R^+} : \mathbb{R}^+ \to S^1$ given by the equation $p|_{R^+}(x) = (\cos 2\pi x, \sin 2\pi x)$ is surjective and a local homeomorphism, but not a covering map. One can see that p is not a covering map because of the behavior of the point $b_0 = p|_{R^+}(0) = (1, 0) \in S^1$. More specifically, the point has no neighbourhood which is evenly covered by the map p. From Example 1.3.5 we know that a usual neighbourhood of the point b_0 in S^1 can be written as $(b_0 - \epsilon, b_0 + \epsilon)$ or depending on p it is $(p|_{R^+}(0) - \epsilon, p|_{R^+}(0) + \epsilon)$. The pre-image of these neighbourhoods is the union of disjoint intervals $(n - \epsilon, n + \epsilon)$, where $n \in \mathbb{Z}$. However, for $n = 0$ it becomes the disjoint union of the interval $(0, \epsilon)$ and intervals $(n - \epsilon, n + \epsilon)$ for $n \in \mathbb{N}$. Each of the intervals of second kind is evenly covered by the map p as in Example 1.3.5, but the interval $(0, \epsilon)$ is not. Therefore, $p|_{R^+}$ is not a covering map.

The preceding example shows that a restriction of a covering map might not be a covering map itself. However, in case of an additional condition, we get the following result.

Theorem 1.3.7. *Let $p : E \to B$ be a covering map. If B_0 is a subspace of B and $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \to B_0$ obtained by restricting p is a covering map.*

Proof. Let $b_0 \in B_0$ and U be an open set in B such that U is evenly covered by p and $b_0 \in U$. Let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Then $U \cap B_0$ is a neighbourhood of b_0 in B_0. Moreover, sets $V_\alpha \cap E_0$ are disjoint open sets in E_0 whose union is equal to $p^{-1}(U \cap B_0)$, where each $V_\alpha \cap E_0$ is mapped homeomorphically onto $U \cap B_0$ by p. Hence, p_0 is a covering map. □

Theorem 1.3.8. *Let $p : E \to B$ and $p' : E' \to B'$ be covering maps. Then $p \times p' : E \times E' \to B \times B'$ is a covering map.*

Proof. Take $b \in B$ and $b' \in B'$, let U and U' be neighbourhoods of b and b', respectively, that are evenly covered by p and p'. Also, let $\{V_\alpha\}$ and $\{V'_\alpha\}$ be partitions of $p^{-1}(U)$ and $p'^{-1}(U')$, respectively, into slices. Then the inverse image under $p \times p'$ of the open set $U \times U'$ is the union of all the sets $V_\alpha \times V'_\alpha$. These are disjoint open sets of $E \times E'$ where each of them is mapped homeomorphically onto $U \times U'$ by the map $p \times p'$. Therefore, $p \times p'$ is a covering map. □

Example 1.3.9. Consider the torus $T_2 = S^1 \times S^1$. Then the product map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, where p is a covering map from Example 1.3.5, is a covering of the torus by the plane $\mathbb{R}^2$. Since we typically think of S^1 as a subset of $\mathbb{R}^2$, this representation of the torus is the subset of $\mathbb{R}^4$. Each of unit squares $[n, n+1] \times [m, m+1]$ gets wrapped by $p \times p$ entirely around the torus.

Example 1.3.10. Consider the covering map $p \times p$ from Example 1.3.9. Let b_0 denote the point $p(0)$ of S^1 and let B_0 denote the subspace $B_0 = (S^1 \times b_0) \cup (b_0 \times S^1) \subset S^1 \times S^1$. Then B_0 is the union of two circles which have the point b_0 in common. This is what we call the **figure-eight space**. Considering the space $E_0 = p^{-1}(B_0)$ which is the infinite grid $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$, the map $p_0 : E_0 \to B_0$ is a covering map of the figure-eight space by Theorem 1.3.7 since it is a restriction of the covering map $p \times p$.

The covering spaces are used to prove a classic result in algebraic topology about the fundamental group of a circle. The idea behind it is that the fundamental group of S^1 is generated by starting at $(1, 0)$ and creating loops that wrap around S^1 a positive integer number of times (counterclockwise) and loops that wrap around S^1 a negative integer number of times (clockwise). For the full proof the reader is referred to [Mun00] or [Wil04].

Theorem 1.3.11. *The fundamental group of S^1 is isomorphic to the additive group of integers.*

Sketch of the proof. Consider a bijection from $\mathbb{R}$ to a helix in $\mathbb{R}^3$ with a parametrisation defined by $(\cos 2\pi s, \sin 2\pi s, s)$. We also identify S^1 as a circle of unit radius inside $\mathbb{R}^2$. Let $p : \mathbb{R} \to S^1$ be a map, which is also a covering map, such that $p(s) = (\cos 2\pi x, \sin 2\pi x)$. This function can be thought of as a projection map from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by $(x, y, z) \mapsto (x, y)$. This means that $\mathbb{R}$ is a covering space of S^1. Consider the map $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$. One can show that this map is a group homomorphism, which gives an isomorphism between $\pi_1(S^1, b_0)$ and $\mathbb{Z}$. □

1.4 Retractions and Deformation Retracts

Definition 1.4.1. If $A \subset X$, a **retraction** of X onto A is a continuous map $r : X \to A$ such that $r|_A$ is the identity map of A. If such a map r exists, we say that A is a **retract** of X.

Lemma 1.4.2. If $a_0 \in A$ and $r : X \to A$ is a retraction, then $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.

Proof. Let $\iota : A \to X$ be the inclusion map. Then $r \circ \iota = \mathbb{1}_A$ by construction. Then

$$r_* \circ \iota_* = (r \circ \iota)_* = \mathbb{1}_{A*} = \mathbb{1}_{\pi_1(A)}$$

by Theorem 1.2.12. Since the right side is an isomorphism r_* has to be a surjection, while ι_* has to be an injection. □

Theorem 1.4.3 (No-retraction Theorem)**.** *There is no retraction of B^2 onto S^1.*

Proof. If S^1 was a retract of B^2, then the homomorphism induced by the inclusion $\iota : S^1 \to B^2$ would be injective. However, the fundamental group of S^1 is non-trivial while the fundamental group of B^2 is trivial. □

Example 1.4.4. There is a retraction r of $\mathbb{R}^2 \backslash \{0\}$ onto S^1 given by equation $r(x) = x/||x||$. Therefore, ι_*, where $\iota : S^1 \to \mathbb{R}^2 \backslash \{0\}$ is the inclusion map, has to be injective, and, hence, non-trivial or, in other words, not nulhomotopic. Similarly, i_*, where $i : S^1 \to S^1$ is the identity map, is the identity homomorphism, and hence non-trivial or not nulhomotopic.

Definition 1.4.5. Let $A \subset X$. We call A a **deformation retract** of X if the identity map of X is homotopic to a map that carries X into A. In other words, there exists a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$, $H(x, 1) \in A$ for all $x \in X$ and $H(a, t) = a$ for all $a \in A$. In this case, we call the homotopy H a **deformation retraction** of X onto A.

Note that the map $r : X \to A$ defined as $r(x) = H(x, 1)$ is a retraction of X onto A, and H is a homotopy between the identity map of X and the map $j \circ r$, where $j : A \to X$ is the inclusion map.

Theorem 1.4.6. *Let A be a deformation retract of X and let $x_0 \in A$. Then the inclusion map $\iota : (A, x_0) \to (X, x_0)$ induces an isomorphism of fundamental groups.*

Proof. Let $r : X \to A$ be the retraction between noted spaces. Then $r \circ \iota$ is the identity map of A, and by Theorem 1.2.12, $r_* \circ \iota_*$ is the identity homomorphism of $\pi_1(A, b_0)$, where $b_0 \in A$.

Consider the composition $\iota \circ r : X \to X$, which maps X to itself, but is not the identity map. It is homotopic to the identity map via a homotopy fixing the points of A, i.e., a homotopy $H : X \times I \to X$ with $H(x, 0) = \iota \circ r(x)$, $H(x, 1) = x$, and $H(x_0, t) = x_0$ for all $t \in I$. By Theorem 1.2 since a deformation retraction gives a base-point preserving homotopy between $\iota \circ r$ and $\mathbb{1}_X$, we have $(\mathbb{1}_X)_* = \iota_* \circ r_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$. We know ι_* is injective. It is also surjective since for any class $[f]$ in $\pi_1(X, x_0)$, we have $[f] = \iota_*(r_*([f]))$. Therefore, it is an isomorphism. □

Example 1.4.7. From the theorem above, one can induce that the inclusion map $\iota : S^n \to \mathbb{R}^{n+1}\backslash\{0\}$ induces an isomorphism of fundamental groups. Thus, $\pi_1(S^n) \cong \pi_1(\mathbb{R}^{n+1}\backslash\{0\})$.

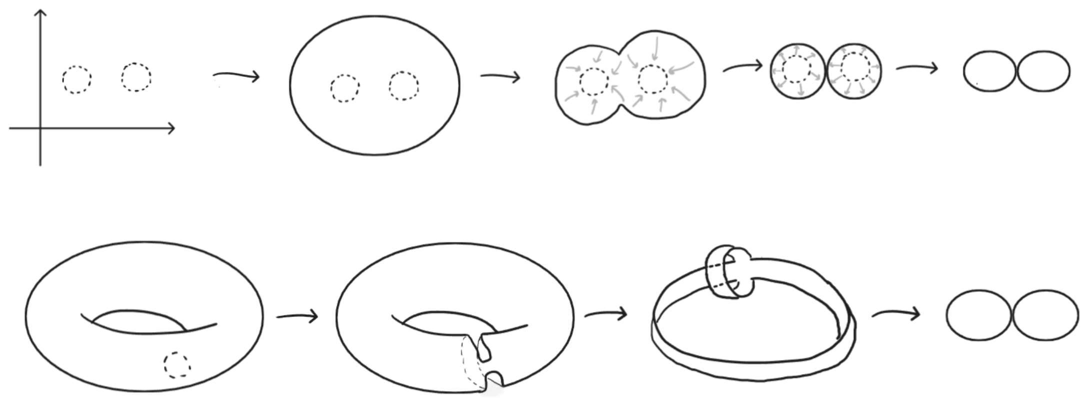


Figure 1.4.1: Deformation retractions following $\mathbb{R}^2\backslash\{p, q\}$ above and the punctured torus below, both resulting in the figure-eight.

Example 1.4.8. Consider $\mathbb{R}^2\backslash\{p, q\}$, where $p, q \in \mathbb{R}^2$, the **doubly punctured plane**, which has the figure-eight (recall Example 1.3.10) as a deformation retract. Another space which has the figure-eight as a deformation retract is the punctured torus, i.e. $T^2\backslash p$ for some point $p \in T^2$. The deformations from this example can be seen in the Figure 1.4.1.

1.5 Homotopy Type

Definition 1.5.1. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. Suppose that the map $g \circ f : X \to X$ is homotopic to the identity map of X, and the map $f \circ g : Y \to Y$ is homotopic to the identity map of Y. Then maps f and g are called **homotopy equivalences**, and each of them is said to be a **homotopy inverse** of the other. Topological spaces X, Y are said to be **homotopy equivalent** or **of the same homotopy type**, where we denote it by $X \simeq Y$, when there are homotopy equivalences between the spaces.

Note that every homeomorphism $f : X \to Y$ is a homotopy equivalence since we can take $g := f^{-1}$. Then if there are spaces X and Y such that $X \cong Y$, it would also mean that $X \simeq Y$. However, the converse of the statement is not true: consider $\mathbb{R}$ and $\{0\}$, which are homotopy equivalent but not homeomorphic. In the Section *2.1.3* we have proved that the relation of path-homotopy equivalence is an equivalence relation. The same can be done for more general type of homotopy to show that the relation of homotopy equivalence is an equivalence relation.

Example 1.5.2. If A is a deformation retract of X, then A has the same homotopy type as X. To show this, take the inclusion map $\iota : A \to X$ and the retraction map $r : X \to A$. Then the composition $r \circ \iota : X \to X$ is the identity map of A, and the composition $\iota \circ r$ is supposed to be homotopic to the identity map of X by the definition of the deformation retraction. With this example, one can think of contractible spaces as spaces that have the homotopy type of a one-point space.

Example 1.5.3. Consider the figure-eight space X and the **theta space**, defined as $\theta = S^1 \cup (0 \times [-1, 1])$. The theta space is also a deformation retract of $\mathbb{R}^2 \backslash \{p, q\}$, but it is not a deformation retract of the figure-eight-space. To see that note that the "bar" $(0 \times [-1, 1])$ in the theta space would need to remain unchanged during the deformation, but it is not a subspace of the figure eight.

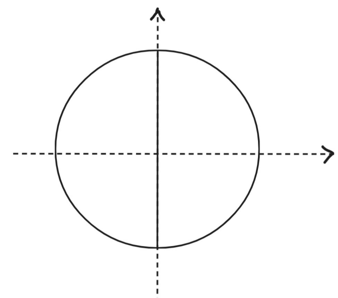


Figure 1.5.1: Theta space.

However, we can describe the homotopy equivalences between them. Consider the figure-eight to be two congruent, tangent circles and the θ space to be a circle with a diameter drawn. Then the map $g : Y \to X$ can be described as contracting the circle along the diameter to the center of the circle. Similarly, the map $f : X \to Y$ can be described as stretching each tangent circle to fit into a half of the θ space.

Note that spaces being homotopy equivalent does not mean that they have isomorphic fundamental groups yet. To show this, we need to look at the case when the base point does not remain the same during the homotopy.

Lemma 1.5.4. Let $h, k : X \to Y$ be continuous maps with $y_0 = h(x_0)$ and $y_1 = k(x_0)$. If h and k are homotopic, then there exists a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$ or, in other words, the following diagram commutes.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, dashed, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

Proof. Let $H : X \times I \to Y$ be the homotopy between h and k. Define the required path α from y_0 to y_1 as $\alpha(t) = H(x_0, t)$. Consider an element $f : I \to X$ of $\pi_1(X, x_0)$, a path c in $X \times I$ given as $c(t) = (x_0, t)$ and loops f_0 and f_1 in the space $X \times I$ given as $f_0(s) = (f(s), 0)$ and $f_1(s) = (f(s), 1)$. Then $H \circ f_0 = h \circ f$ and $H \circ f_1 = k \circ f$, while $H \circ c = \alpha$.
Consider a map $F : I \times I \to X \times I$, such that $F(s, t) = (f(s), t)$ and the following paths in $I \times I$, which run along the four edges of $I \times I$:

$$\beta_0(s) = (s, 0) \text{ and } \beta_1(s) = (s, 1),$$

$$\gamma_0(t) = (0, t) \text{ and } \gamma_1(t) = (1, t).$$

Then $F \circ \beta_0 = f_0$ and $F \circ \beta_1 = f_1$, while $F \circ \gamma_0 = F \circ \gamma_1 = c$.
The broken-line paths $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are both paths in $I \times I$ from $(0, 0)$ to $(1, 1)$ and since $I \times I$ is convex, there is a path homotopy between them by Example 1.1.6. Then $F \circ G$ is a path homotopy in $X \times I$ between $f_0 * c$ and $c * f_1$. Therefore, $H \circ (F \circ G)$ is a path homotopy in Y between

$$(H \circ f_0) * (H \circ c)) = (h \circ f) * \alpha \qquad \text{and} \qquad (H \circ c) * (H \circ f_1) = \alpha * (k \circ f),$$

which would mean that

$$[k \circ f] = [\overline{\alpha}] * [h \circ f] * [\alpha]$$

or

$$k_*([f]) = \hat{\alpha}(h_*([f])),$$

as needed. ∎

The immediate consequence of the Lemma above is that in case of h_* being injective, surjective or trivial, k_* has the same property. Moreover, if $h : X \to Y$ is nulhomotopic, then h_* is the trivial homomorphism. The most important result of this lemma allows us to extend the idea of fundamental group to spaces of the same homotopy type.

Theorem 1.5.5. *Let $f : (X, x_0) \to (Y, y_0)$ be a continuous map. If f is a homotopy equivalence then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Proof. Consider $g : Y \to X$ be a homotopy inverse for f and maps

$$(X, x_0) \xrightarrow{f_{x_0}} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f_{x_1}} (Y, y_1),$$

where $x_1 = g(y_0)$ and $y_1 = f(x_1)$. Then we have induced homomorphisms as follows:

$$\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1).$$

By assumption, $g \circ f : (X, x_0) \to (X, x_1)$ is homotopic to the identity map, so there is a path α in X such that $(g \circ f)_* = \hat{\alpha} \circ (\mathbb{1}_X)_* = \hat{\alpha}$. It follows that $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism and g_* is surjective. Similarly, since $f \circ g$ is homotopic to the identity map $\mathbb{1}_Y$, the homomorphism $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism and g_* is injective. Therefore, g_* is an isomorphism. Moreover, we can conclude that $(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}$ and, thus, $(f_{x_0})_*$ is also an isomorphism. ∎

1.6 Fundamental Groups of Other Surfaces

Theorem 1.6.1. *Suppose $X = U \cup V$, where both U and V are open sets of X. Suppose $U \cap V$ is path-connected, and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphisms*

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \text{ and } j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$. In other words, given any loop f in X based at x_0, it is path homotopic to a product of the form $g_1 * g_2 * \ldots * g_n$, where each g_i is a loop in X based at x_0 which lies entirely either in U or V.

Proof. First, use the Lebesgue number Lemma A.0.30 to choose a subdivision $\{b_i\}$ of I such that for all i the set $f([b_{i-1}, b_i])$ is contained in either U or V. If for all i, the set $f([b_{i-1}, b_i])$ is contained in $U \cap V$, pick this division. Otherwise, let i be an index such that $f(b_i) \notin U \cap V$. Both of the sets $f([b_{i-1}, b_i])$ and $f([b_i, b_{i+1}])$ lie fully in either U or V. If $f(b_i) \in U$ then both of the sets must lie in U, otherwise, they both must belong to V. In both cases, consider the same division of I but without b_i - let's call it $\{c_i\}$. This subdivision satisfies the main condition - for all i the set $f([c_{i-1}, c_i])$ belongs to either U or V - therefore, we can do this operation until we reach the desired subdivision. Let $\{a_i\}$ be the subdivision of I obtained, i.e., for all i we have $f([a_{i-1}, a_i])$ is either in U or V and $f(a_i) \in U \cap V$.
Now, define f_i to be the path in X that equals the linear map of I onto $[a_{i-1}, a_i]$ followed by f. Then f_i is a path that lies either in U or V, and $[f] = [f_1] * \ldots * [f_n]$. For each i, since $U \cap V$ is path-connected, we can choose a path α_i in $U \cap V$ from x_0 to $f(a_i)$. Since $f(a_0) = f(a_n) = x_0$, we can choose α_0 and α_n to be constant paths at x_0. Now for each i we have $g_i = \alpha_{i-1} * f_i * \overline{\alpha_i}$. This means that g_i is a loop in X based at x_0 whose image lies either in U or in V. Then we have

$$[g_1] * \ldots * [g_n] = [\alpha_0] * [f_0] * [\overline{\alpha_1}] * [\alpha_1] * [f_1] * [\overline{\alpha_2}] * \ldots * [\alpha_{n-1}] * [f_n] * [\overline{\alpha_n}] =$$

$$[\alpha_0] * [f_1] * \ldots * [f_n] * [\alpha_n] = [f_1] * \ldots * [f_n],$$

as needed. □

Corollary 1.6.2. Suppose $X = U \cup V$, where both U and V are open sets of X. Suppose $U \cap V$ is path-connected and non-empty. If U and V are simply-connected, then X is simply-connected.

Proof. Since $U \cap V$ is non-empty, there exists a point $x_0 \in U \cap V$. Both U and V are simply-connected, so $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ are trivial. Then both of the induced homomorphisms of the inclusion mappings i_* and j_* are trivial homomorphisms, and, thus, $\pi_1(X, x_0)$ is trivial. □

Theorem 1.6.3. *The n-sphere S^n is simply-connected for $n \geqslant 2$.*

Proof. First, note that for $n \geqslant 1$, the punctured sphere $S^n \backslash \{p\}$ is homeomorphic to $\mathbb{R}^n$, since we can define the stereographic projection as a homeomorphism between them. Firstly, let's show that for $n \geqslant 1$, the punctured sphere $S^n \backslash \{p\}$ is homeomorphic to $\mathbb{R}^n$. For a point $p = (0, \ldots, 0, 1) \in S^n$ define a map $f : (S^n \backslash \{p\}) \to \mathbb{R}^n$ as stereographic projection:

$$f(x) = f(x_1, x_2, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n).$$

To show that this map is a homeomorphism, we can check that the map $g : \mathbb{R}^n \to (S^n \backslash \{p\})$, defined by

$$g(y) = g(y_1, \ldots, y_n) = (t(y)y_1, \ldots, t(y)y_n, 1 - t(y)),$$

where $t(y) = 2/(1 + ||y||^2)$, is both right and left inverse for f. Another way of thinking about it is understanding what it is doing: if we take a line passing through p and the point

$x \in (S^n \backslash \{p\})$, it would intersect the plane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ in only one point $f(x) \times \{0\}$. Note that the reflection map $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n, -x_{n+1})$ defines a homeomorphism of $S^n \backslash p$ with $S^n \backslash \{q\}$, where $q = (0, \ldots, 0, -1) \in S^n$ is the south pole of the sphere, so the latter space is also homeomorphic to $\mathbb{R}^n$.

Now take $U = S^n \backslash \{p\}$ and $V = S^n \backslash \{q\}$ be open sets of S^n. For $n \geqslant 1$, the sphere S^n is path-connected since both $U \cong \mathbb{R}^n$ and $V \cong \mathbb{R}^n$ are path-connected and have the point $(1, 0, \ldots, 0)$ in common. To show that S^n is simply-connected, note that $U \cap V = S^n \backslash \{p, q\}$ which is homeomorphic to $\mathbb{R}^n \backslash \{0\}$. The latter space is path-connected and, thus, $U \cap V$ is path-connected. Therefore, by Corollary 1.6.2, $U \cup V = S^n$ is simply-connected. ☐

Definition 1.6.4. A topological space M is a **topological manifold** of dimension n (or **topological n-manifold**) if

- M is Hausdorff (recall A.0.24),
- M is second-countable (recall A.0.25), and
- M is locally Euclidean: for all points $m \in M$ there exists an open neighbourhood in which is homeomorphic to an open subset of $\mathbb{R}^n$.

A topological 2-manifold is called a **surface**.

Theorem 1.6.5. *$\pi_1(X \times Y, x_0 \times y_0)$ is isomorphic with $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection maps. Using the induced homomorphisms of given maps, define a homomorphism

$$\Phi : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

by the equation

$$\Phi([f]) = \big(p_*([f]), q_*([f])\big) = \big([p \circ f], [q \circ f]\big).$$

To show that the map Φ is an isomorphism we need to show that it is bijective. To show that the map is surjective, let $g : U \to X$ be a loop based at x_0 and let $h : I \to Y$ be a loop based at y_0. Also, define $f : I \to X \times Y$ such that $f(s) = g(s) \times h(s)$. Then f is a loop in $X \times Y$ based at $x_0 \times y_0$ with

$$\Phi([f]) = ([p \circ f], [q \circ f]) = ([g], [h]),$$

which means that the element $([g], [h])$ lies in the image of Φ. More intuitively, if f is a loop based at (x_0, y_0), it is nothing more than a pair of loops in X and Y based at x_0 and y_0. Similarly, homotopies of loops are nothing but pairs of homotopies of pairs of loops.

To show that Φ is one-to-one, define $f : I \to X \times Y$ as a loop in $X \times Y$ based at $x_0 \times y_0$ with an identity element being $\Phi([f]) = \big([p \circ f], [q \circ f]\big)$, which means that $p \circ f \simeq_p e_{x_0}$ and $q \circ f \simeq_p e_{y_0}$. Let G and H be the respective homotopies in X and Y. Then the map $F : I \times I \to X \times Y$ defined by $F(s,t) = G(s,t) \times H(s,t)$ is a path homotopy between f and a constant loop based at $x_0 \times y_0$. ☐

Note that the preceding theorem can be extended to a finite product of spaces. Moreover, if any of the spaces end up being contractible, they can be dropped from the product.

Example 1.6.6. A natural example to consider, given that $\pi_1(S^1) \cong \mathbb{Z}$, is the torus $T^2 = S^1 \times S^1$. Then $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$.

Chapter 2

Free Groups

So far in the previous sections we were able to compute the fundamental group in some basic cases. For more complicated cases, we need to develop a few more strategies and skills to be able to describe the structure of the group itself.

Recall the definition of the **direct product** $G = G_1 \times G_2 \times \ldots \times G_n$ of a finite number of groups $\{G_i\}_{i=1}^n$. The elements of G are ordered n-tuples $g = (g_1, \ldots, g_n)$, where $g_i \in G_i$, with the operation of multiplication denoted by

$$(g_1, \ldots, g_n)(h_1, \ldots, h_n) = (g_1 h_1, \ldots, g_n h_n).$$

This idea can be extended to a case with infinitely many groups: consider an infinite collection of groups $\{G_i\}_{i \in I}$, where I is an index set. The direct product in this case is defined as $\prod_{i \in I} G_i$. Its elements are functions which assign to each index $i \in I$ an element $g_i \in G_i$ with the similar definition for the multiplication.

2.1 Free Groups

Given a non-empty set X, we would like to construct a free group on this set. There are different ways to describe free groups and products, and we are going to follow the idea from [Hun12]. If $X = \varnothing$, then the free group is going to be the trivial group $\langle e \rangle$. Otherwise, let X^{-1} be a set disjoint from X such that $|X| = |X^{-1}|$. Choose a bijection $X \to X^{-1}$ and denote an image of $x \in X$ by x^{-1}. Choose an element $\mathbb{I}$ disjoint from $X \cup X^{-1}$.

Definition 2.1.1. In this context, a **word** on X is a sequence $(a_1, a_2, \ldots)$ such that $a_j \in X$ for $j \in \mathbb{N}$ and for some $n \in \mathbb{N}$, $a_k = 1$ for all integer $k \geqslant n$. Define $\mathbb{I} = (1, 1, 1, \ldots)$ to be the **empty word**.

There are infinitely many such words we can construct on a set, although, some of them seem to be equivalent. To deal with this problem there are a few **reduction** operations we can do to get the **reduced** word:

1. if we have adjacent x and x^{-1}, we can delete both,

2. if $a_k = 1$ for some $k \in \mathbb{N}$, then $a_i = 1$ for all $i \geqslant k$.

Notice that every reduced word is of the form $(x_1^{\lambda_1}, x_2^{\lambda_2}, \dots, x_n^{\lambda_n}, 1, 1, \dots)$, where $x_i \in X$ and $\lambda_i = \pm 1$. We will denote such word by $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$. For simplicity, one can also combine the adjacent identical elements x and x to write x^2 and so on for higher powers.

Example 2.1.2. The empty word $\mathbb{I}$ is reduced without any reduced operations applied. Consider a set $X = \{x, y, z\}$. Let $w_1 = (x, x)$ and $w_2 = (x^{-1}, y, y, y, x^{-1}, x^{-1}, x, z, z^{-1})$ be words. Their juxtaposition is the sequence $w = (x, x, x^{-1}, y, y, y, x^{-1}, x^{-1}, x, z, z^{-1})$, which can be reduced to $(x, y, y, y, x^{-1}) = xy^3x^{-1}$.

Two reduced words $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$ and $y_1^{\delta_1}, y_2^{\delta_2}, \dots, y_m^{\delta_m}$ with $\lambda_i, \delta_j = \pm 1$ are equal if and only if both are $\mathbb{I}$ or $m = n$ and for all $1 \leqslant i \leqslant n$ we have $x_i = y_i$ and $\lambda_i = \delta_i$. With this definition denote the set of all reduced words on a set X as $F(X)$.
To make this a group we need to add an identity and a binary operation to it.

Definition 2.1.3. Consider **juxtaposition** of two words

$$x * y = (x_1^{\lambda_1}, x_2^{\lambda_2}, \dots, x_n^{\lambda_n}) * (y_1^{\delta_1}, y_2^{\delta_2}, \dots, y_m^{\delta_m}) = (x_1^{\lambda_1}, x_2^{\lambda_2}, \dots, x_n^{\lambda_n}, y_1^{\delta_1}, y_2^{\delta_2}, \dots, y_m^{\delta_m}),$$

both taken on a given set X.

Intuitively, the empty word $\mathbb{I}$ behaves like an identity element, i.e., $\mathbb{I} * w = w * \mathbb{I} = w$, for any non-empty word $w \in F(X)$. Also, note that the juxtaposition of two reduced words might not be reduced, but one can reduce it using the reduction operations.

Theorem 2.1.4. *If X is a nonempty set and $F = F(X)$ is the set of all reduced words on X, then F is a* ***free group*** *under juxtaposition and it is denoted by $F = \langle X \rangle$ instead.*

Proof. To verify that F is a group we need to check all properties of a group. We know that the empty word is an identity and a word $(x_1^{\lambda_1}, x_2^{\lambda_2}, \dots, x_n^{\lambda_n})$ has an inverse $(x_n^{-\lambda_n}, x_{n-1}^{-\lambda_{n-1}}, \dots, x_1^{-\lambda_1})$. To verify associativity, note that we do not need to reduce the juxtaposition until the very end. With this, one gets the products of three reduced words $x, y, z \in X$ equal to

$$\begin{aligned}
(x * y) * z &= ((x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}) * (y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m})) * (z_1^{\gamma_1} z_2^{\gamma_2} \dots z_k^{\gamma_k}) \\
&= (x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m}) * (z_1^{\gamma_1} z_2^{\gamma_2} \dots z_k^{\gamma_k}) \\
&= x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m} z_1^{\gamma_1} z_2^{\gamma_2} \dots z_k^{\gamma_k} \\
&= (x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}) * ((y_1^{\delta_1} y_2^{\delta_2} \dots y_m^{\delta_m}) * (z_1^{\gamma_1} z_2^{\gamma_2} \dots z_k^{\gamma_k})) = x * (y * z).
\end{aligned}$$

□

Here, we mention some properties of free groups. If $|X| \geqslant 2$, then the free group on X is not abelian since for $x, y \in X$, such that $x \neq y$, we have words xy and yx being both reduced but not equal to each other. Also, every element of such a group except the identity has infinite order. This being said, if $X = \{a\}$, then F is an infinite cyclic group.

Theorem 2.1.5 (Universal Mapping Property)**.** *Let F be the free group on a set X and $\iota : X \to F$ an inclusion map. If G is a group and $\phi : X \to G$ a map of sets, then there exists a unique homomorphism of groups $\overline{\phi} : F \to G$ such that $\overline{\phi} \circ \iota = \phi$, i.e., the following diagram commutes.*

$$\begin{tikzcd} F \arrow[dr, dashed, "\overline{\phi}"] & \\ X \arrow[u, hook, "\iota"] \arrow[r, "\phi"'] & G \end{tikzcd}$$

Proof. Define $\overline{\phi}(\mathbb{1}) = e$ and for a non-empty reduced word on X, define

$$\overline{\phi}(x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}) = \phi(x_1)^{\lambda_1} \phi(x_2)^{\lambda_2} \dots \phi(x_n)^{\lambda_n}.$$

Since G is a group and $\lambda_i = \pm 1$ for all $1 \leqslant i \leqslant n$, the product above is well-defined in G. Such a definition of $\overline{\phi}$ automatically results in $\overline{\phi}$ being a homomorphism such that $\overline{\phi} \circ \iota = \phi$. To show that $\overline{\phi}$ is indeed unique, consider a homomorphism $g : F \to G$ such that $g \circ \iota = \phi$. Then

$$\begin{aligned} g(x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}) &= g(x_1)^{\lambda_1} g(x_2)^{\lambda_2} \dots g(x_n)^{\lambda_n} \\ &= g(\iota(x_1)^{\lambda_1}) g(\iota(x_2)^{\lambda_2}) \dots g(\iota(x_n)^{\lambda_n}) \\ &= \phi(x_1)^{\lambda_1} \phi(x_2)^{\lambda_2} \dots \phi(x_n)^{\lambda_n} = \overline{\phi}(x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}), \end{aligned}$$

which means $\overline{\phi}$ is unique. □

The theorem above shows that F is a free object on the set X in the category of groups according to the Definition B.0.15. This being said, if F' is another free object on the same set X with $\lambda : X \to F'$ in the category of groups, then there is an isomorphism $\phi : F \to F'$ such that $\phi \circ \iota = \lambda$.

Corollary 2.1.6. Every group G is the homomorphic image of a free group.

The free group on X is also said to be the *freest* group generated on a set X. To see why note that in an arbitrary group there are different products of elements, which give an identity element as a result. For example,

1. $x * x^{-1} = e$ for any element of any group;
2. in a cyclic group of order n, $x^n = e$.

Any such product is called a **relation** on a group X. Relations of type (1), which come from properties of a group, are said to be **trivial**, while all other ones, like type (2), are said to be **non-trivial**. The other way to define a free group on a set is to take a set X with only trivial relations between its elements.

Definition 2.1.7. Let X be a set and R be a set of reduced words on X. A group G is said to be the **group defined by the generators** $x \in X$ **and relations** $w = e$ for $w \in R$ provided $G \cong F/N$, where F is a free group on X and N the normal subgroup of F generated by R. In this case, we call $G = \langle X | R \rangle$ a **presentation** of G.

These notions also lead to the idea that one can completely describe a group G with its generating set X and relations set R between them. Note that a presentation of a group is not unique. To see this, consider a cyclic group $\mathbb{Z}_6$ with presentations $\langle a|a^6\rangle$ and $\langle a,b|a^2 = b^3 = a^{-1}b^{-1}ab\rangle$.
There is a relation between free groups and free abelian groups. For that recall that if x and y are elements of a group G, then the element $[x,y] = xyx^{-1}y^{-1} \in G$ is called the **commutator** of x and y. The notation $[G,G]$ denotes the subgroup of G generated by all commutators - the **commutator subgroup**. Commutators are, in a sense, a measure how much of G fails to be commutative. In particular, the commutator subgroup is trivial if and only if all commutators are the identities. We know a few facts about this subgroup:

Theorem 2.1.8. *Given a group G, the commutator subgroup $[G,G]$ is a normal subgroup and the quotient group $G/[G,G]$ is abelian. Moreover, if $h : G \to H$ is a homomorphism from G to an abelian group H, then the kernel of h contains $[G,G]$, and hence h induces a homomorphism $k : G/[G,G] \to H$.*

Proof. The theorem consists of 3 different facts, each of which is going to proved in a separate step.

Step 1. To show that $[G,G]$ is normal, first, we need to show that any conjugate of a commutator is in $[G,G]$ as well:

$$\begin{aligned} g[x,y]g^{-1} &= g(xyx^{-1}y^{-1})g^{-1} \\ &= (gxyx^{-1})(y^{-1}g^{-1}) \\ &= (gxyx^{-1})(g^{-1}y^{-1}yg)(y^{-1}g^{-1}) \\ &= ((gx)y(gx)^{-1}y^{-1})(ygy^{-1}g^{-1}) \\ &= [gx,y][y,g], \end{aligned}$$

which is known to be in $[G,G]$. Now, consider an arbitrary element z of $[G,G]$. This element is a product of commutators and their inverses. Since

$$[x,y]^{-1} = (xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1} = [y,x]$$

or, in other words, every inverse of a commutator is a commutator itself, z is just a product of commutators $z_1, z_2, \ldots, z_n$. Then its conjugate is

$$gzg^{-1} = gz_1z_2\ldots z_ng^{-1} = gz_1(g^{-1}g)z_2(g^{-1}\ldots g)x_ng^{-1} = (gz_1g^{-1})(gz_2g^{-1})\ldots(gx_ng^{-1}),$$

which is a product of commutators' conjugates, which we know is in $[G,G]$, as well. Therefore, the group $[G,G]$ is normal.

Step 2. For this step let $G' = [G,G]$. To show that $G/[G,G]$ is abelian, we need

$$(aG')(bG') = (bG')(aG'),$$

which is equivalent to

$$abG' = baG',$$

which is equivalent to

$$a^{-1}b^{-1}abG' = G'.$$

Since $a^{-1}b^{-1}ab$ is a commutator on its own, it belongs to G' and the last statement follows.

Step 3. Since H is abelian by assumption, h carries each commutator to the identity element of H. Hence, the kernel of h contains the whole commutator subgroup $[G, G]$, so h induced the desired homomorphism k. $\square$

Hence, using this we can transform any free group F to a free abelian group $F/[F, F]$, which is called the **abelianization**, using the natural projection $\pi : F \to F/[F, F]$.

Definition 2.1.9. If G is a free abelian group, the **rank** of G is the number of elements in the generating set of G.

Since for any free group G with n generators the free abelian group $G/[G, G]$ has rank n, any system of free generators for G would have n elements.

Theorem 2.1.10. *If F and F' are free groups on finite sets S and S', then F and F' are isomorphic if and only if S and S' have the same rank.*

Since now we are dealing with finitely generated abelian groups, we need some properties of such groups. First, recall that the set of all elements of an arbitrary abelian group A that have finite order is called the **torsion subgroup**. If we denote the torsion subgroup by T, then the quotient group A/T is going to be torsion free. In case groups A and A' are isomorphic, their torsion subgroups and quotients mod torsion subgroups are also isomorphic. The converse is true, however, only for finitely generated abelian groups.

Theorem 2.1.11 ([Mas91],[Hun12])**.** *Consider only finitely generated abelian groups. Then we have the following:*

1. *Let A be a finitely generated abelian group and let T be its torsion subgroup. Then, T and A/T are also finitely generated, and A is isomorphic to the direct product $T \times A/T$. Hence, the structure or A is completely determined by its torsion subgroup T and its torsion-free subgroup A/T.*
2. *Every finitely generated torsion-free abelian group is a free abelian group of finite rank.*
3. *Every finitely generated abelian group G is isomorphic to a product*

$$\mathbb{Z}_{d_1} \oplus \ldots \oplus Z_{d_k} \oplus \mathbb{Z}^{\oplus n} \cong (\mathbb{Z}/d_1\mathbb{Z}) \oplus \ldots \oplus (\mathbb{Z}/d_k\mathbb{Z}) \oplus \mathbb{Z}^{\oplus n},$$

where $\mathbb{Z}^{\oplus n}$ means the direct product of n copies of the group $\mathbb{Z}$. Moreover, k, n and d_i are all uniquely determined and they completely determine the structure of the group G.

However, how do we extract information about a group from its presentation? This question is answered in, for instance, [D L97]. The reader is welcome to familiarize themselves with the topic, but here we are going to state the needed theorem and use of it for the finite

X. Consider a presentation $P = \langle X|R\rangle$ of a group G with abelianization G_{ab}. Let us fix the notation

$$X = \{x_1, x_2, \ldots, x_r\}, C = \{[x_i, x_j]|1 \leqslant i < j \leqslant r\}, r \in \mathbb{N},$$

where C might be regarded as a subset of any group presented on generators X.

Proposition 2.1.12 ([D L97])**.** If $G = \langle X|R\rangle$, then $G_{ab} = \langle X|R, C\rangle$.

Now, in terms of presentations, part (3) of Theorem 2.1.11 means that every such group G has a unique presentation of the form

$$\langle x_1, \ldots, x_r | x_1^{d_1}, \ldots, x_k^{d_k}, C\rangle,$$

where $k \leqslant r$ and the d_i satisfy the conditions of the theorem.

2.2 Free Product

With the idea from the previous section, one can define the free product of groups. Given a family of mutually disjoint groups $\{G_i|i \in I\}$, let $X = \bigcup_{i\in I} G_i$ and let $\mathbb{I}$ be a one-element set disjoint from X.

Definition 2.2.1. A **word** is a sequence $(a_1, a_2, \ldots)$ such that $a_i \in X \cup \mathbb{I}$ and for some $n \in \mathbb{N}$, $a_i = 1$ for all $i \geqslant n$.

A word in this case also can get **reduced**:

1. if $a_i \in X$ is the identity element of some G_j, then we can delete a_i,

2. if a_i and a_{i+1} belong to the same G_j, we can substitute it with their composition $a_i *_{G_j} a_{i+1}$, and

3. if $a_k = 1$, then $a_i = 1$ for all $i \geqslant k$.

With this reduction operation, the empty word $\mathbb{I}$, represented by the sequence $(1, 1, \ldots, 1)$, is already reduced. Every non-empty reduced word can also be written uniquely as $a_1a_2\ldots a_n = (a_1, a_2, \ldots, a_n, 1, 1, \ldots)$, where $a_i \in X$. Considering the same binary operation, the juxtaposition, we are able to define the set of all reduced words on X and denote it by $\prod^*_{i\in I} G_i$.

Theorem 2.2.2. *$\prod^*_{i\in I} G_i$ forms a group,* ***free product*** *of the family $\{G_i|i \in I\}$, under the juxtaposition.*

We can identify G_i with its isomorphic image in $\prod^*_{i\in I} G_i$.

Theorem 2.2.3 (Characteristic Property of Free Product)**.** *Let $\{G_i|i \in I\}$ be a family of groups with free product $\prod^*_{i\in I} G_i$ and family of inclusions $\iota_i : G_i \to \prod^*_{i\in I} G_i$. If $\{\psi_i : G_i \to H|i \in I\}$ is a family of group homomorphisms onto a group H, then there exists a unique homomorphism $\psi : \prod^*_{i\in I} G_i \to H$ such that $\psi \circ \iota_i = \psi_i$ for all $i \in I$.*

Similarly to the free groups, the free product of groups represents the coproduct in the category of groups.

Theorem 2.2.4. *Consider a group $G = \prod_{i\in I}^{*} G_i$, where all G_i are free groups with $\{a_\alpha\}_{\alpha\in J_i}$ as respective systems of free generators with $\bigcap_{i\in I} J_i = \varnothing$. Then G is a free group with $\{a_\alpha\}_{\alpha\in\bigcup_{i\in I} J_i}$ as a system of free generators.*

The theorem above can be extended to a free product of any finite number of free groups.

Example 2.2.5. If G is a group defined by generators a, b and relations $a^2 = \mathbb{I}$ and $b^3 = \mathbb{I}$, then $G \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Generally, the group defined by the generator c and the relation $c^m = \mathbb{I}$ for some $m \in \mathbb{N}$ is the cyclic group $\mathbb{Z}_m$.

2.3 The Seifert-van Kampen Theorem

In this section, let $X = U \cup V$ be a topological space, where both U and V are open in X. Moreover, suppose that X, U, V, and $U \cap V$ are all path-connected and that the fundamental groups of U and V are known. There are two versions of the main theorem in this section.

Theorem 2.3.1 (Seifert-van Kampen Theorem, modern version)**.** *Let $x_0 \in U \cap V$ and let $\phi_1 : \pi_1(U, x_0) \to H$ and $\phi_2 : \pi_1(V, x_0) \to H$ be homomorphisms. Let i_1, i_2, j_1, j_2 be the homomorphisms indicated below, each induced by inclusion.*

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, dashed, "\Phi"] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

If $\phi_1 \circ i_1 = \phi_2 \circ i_2$, then there exists a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$.

Proof. First, we will show uniqueness of Φ. By Theorem 1.6.1 $\pi_1(X, x_0)$ is generated by the images of the induced homomorphisms j_1 and j_2. Because Φ is determined by ϕ_1 and ϕ_2 on these images, it follows that it is determined on every product of the elements from these images. However, these products include all of the elements, and so Φ is determined by ϕ_1 and ϕ_2 and, therefore, is unique.
To show the existence, consider a path f in X together with its path-homotopy class $[f]$ in X. If f lies in U, V or $U \cap V$, let $[f]_U$, $[f]_V$ and $[f]_{U\cap V}$ denote its path-homotopy class in U, V and $U \cap V$, respectively. The plan is to define several different maps, each building on the previous, and, for that, consider the steps below.
Step 1. Let's define a map ρ which assigns an element of the group H to each loop f based

at x_0 that lies in U or in V. In other words, we want to extend both ϕ_1 and ϕ_2 to a set map ρ defined on all loops in X, which are contained in either U or V. Define an element of the group H by

$$\rho(f) = \begin{cases} \phi_1([f]_U) & \text{if } f \text{ lies in } U, \\ \phi_2([f]_V) & \text{if } f \text{ lies in } V. \end{cases}$$

Note that ρ is well-defined because for f lying in both U and V we have $\phi_1([f]_U) = \phi_1(i_1([f]_{U\cap V}))$ and $\phi_2([f]_V) = \phi_2(i_2([f]_{U\cap V}))$. Moreover, since by assumption $\phi_1 \circ i_1 = \phi_2 \circ i_2$, we also have $\phi_1([f]_U) = \phi_2([f]_V)$. This makes up two facts about our map ρ:

1. If $[f]_U = [g]_U$ or $[f]_V = [g]_V$, then $\rho(f) = \rho(g)$ (by the initial definition of ρ).
2. If both f and g lie in U or both of them lie in V, then $\rho(f * g) = \rho(f) *_H \rho(g)$ since ϕ_1 and ϕ_2 are homomorphisms.

Step 2. Let's extend ρ to a map σ, which assigns an element of H to each path f lying in U or in V such that the map σ also satisfies the condition (1) of ρ and condition (2), when possible, i.e., when $f * g$ is defined. This makes any path be workable as any other closed loop.
For each $x \in X$, choose a path α_x from x_0 to x as follows:

- If $x = x_0$, let α_x be a constant path at x_0.
- If $x \in U \cap V$, let α_x be a path in $U \cap V$.
- If $x \in U$ or $x \in V$ with $x \notin U \cap V$, let α_x be a path in U or V, respectively.

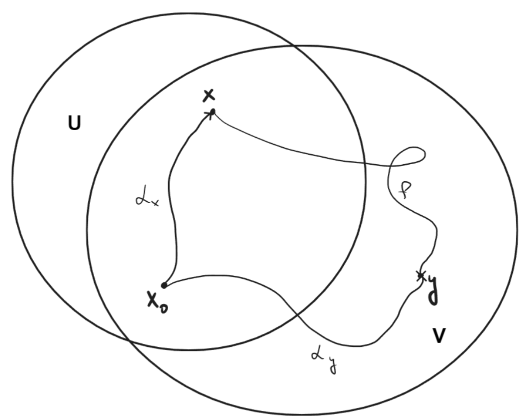


Figure 2.3.0: Construction of a loop.

This way for any path f in U or in V from x to y, we define a loop $L(f)$ based at x_0 such that

$$L(f) = \alpha_x * (f * \overline{\alpha_y}).$$

Note that because of our choice of α_x and α_y, if f was a path in U, then $L(f)$ would be a loop in U as shown on Figure 2.3.0. The same follows if f was a path in V.
Now, define $\sigma(f) = \rho(L(f))$. To show that this map works for us, we need to show that σ is indeed an extension of ρ and that the properties given hold. If f is a loop based at x_0 lying in either U or V, then we have

$$L(f) = \alpha_{x_0} * (f * \overline{\alpha_{x_0}}),$$

where α_{x_0} is a constant path at x_0. Then $L(f)$ is path-homotopic to f in either U or V, so $\rho(L(f)) = \rho(f)$ by property (1) of ρ. Hence, $\sigma(f) = \rho(f)$. To check condition (1), let f and g be paths which are path homotopic in U or V. If F is a path homotopy in U from f to g, then the homotopy $L(F)$ is a path homotopy in U from $L(f)$ to $L(g)$. Thus, $L(f)$ and $L(g)$ are path-homotopic in U and so the condition (1) applies. The same can be done for the case when f and g are path homotopic in V. To check condition (2), let f and g be arbitrary paths in U or V such that $f(0) = x$, $f(1) = g(0) = y$ and $g(1) = z$ so $f * g$ is well-defined. Then we have

$$L(f) * L(g) = (\alpha_x * (f * \overline{\alpha_y})) * (\alpha_y * (g * \overline{\alpha_z})) \simeq_p \alpha_x * ((f * g) * \overline{\alpha_z}),$$

which means that $L(f) * L(g)$ is path homotopic to $L(f * g)$. Therefore,

$$\rho(L(f * g)) = \rho(L(f) * L(g)) = \rho(L(f)) *_H \rho(L(g))$$

by condition (2) for ρ. Hence, $\sigma(f * g) = \sigma(f) *_H \sigma(g)$ and so property (2) is satisfied.
Step 3. Finally, let's extend σ to a set map τ which assigns an element of H to an arbitrary path f of X. Given any path f in X, using the Lebesgue Number Lemma A.0.30, we can choose a subdivision $0 = s_0 < s_1 < \ldots < s_n = 1$ of the interval I such that f maps each of the sub-intervals $[s_{i-1}, s_i]$ into U or V. Let f_i denote the path obtained by restricting f to the sub-interval $[s_{i-1}, s_i]$. Then f_i is a path in U or V with $[f] = [f_1] * \ldots * [f_n]$. Define τ as

$$\tau(f) = \sigma(f_1) *_H \ldots *_H \sigma(f_n).$$

This map will satisfy the similar conditions to ρ and σ:

1. If $[f] = [g]$, then $\tau(f) = \tau(g)$.
2. If $f * g$ is well-defined, then $\tau(f * g) = \tau(f) *_H \tau(g)$.

But before we show that the map actually satisfies these claims, let's show that this definition of τ is actually independent of the choice of subdivision. For this we need to show that the value of $\tau(f)$ remains the same if we add one additional point p to the subdivision. Let i be the index such that $s_{i-1} < p < s_i$ with p being a new point. If we compute $\tau(f)$ using the new subdivision, the only change in the the value is the change of $\sigma(f_i)$ to $\sigma(f_i') *_H \sigma(f_i'')$, where f_i' and f_i'' are paths obtained by restricting f to $[s_{i-1}, p]$ and $[p, s_i]$, respectively. However, since f_i is path homotopic to $f_i' * f_i''$ in U or V, we have $\tau(f_i) = \tau(f_i') *_H \tau(f_i'')$ by conditions (1) and (2) which we know work for τ. Therefore, τ is indeed independent of our choice of subdivision and hence well-defined.
It immediately follows that τ is an extension of σ: if f is already in U or V, then we can use the trivial partition $[0, 1] = \{\{0\}, (0, 1), \{1\}\}$ to define $\tau(f)$ and so $\tau(f) = \sigma(f)$ by definition.
Now, let's show that τ satisfies the condition (1): if $[f] = [g]$, then $\tau(f) = \tau(g)$. Let f and g be paths in X from x to y and let F be the path homotopy between them. Using the compactness of $[0, 1]^2$ for the Lebesgue number lemma A.0.30, we can choose subdivisions $s_0 < \ldots < s_n$ and $t_0 < \ldots < t_m$ of $[0, 1]$ such that F maps each sub-rectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into U or V. Let f_j be the path $f_j(s) = F(s, t_j)$. Then $f_0 = f$ and $f_m = g$. Note that for all pairs of paths f_{j-1} and f_j there exists a subdivision $s_0, \ldots, s_n$ of I such that F carries each rectangle $R_i = [s_{i-1}, s_i] \times [0, 1]$ into either U or V. Given i, consider the linear map of I onto $[s_{i-1}, s_i]$ followed by f or by g - let's call these maps f_i and g_i, respectively. The restriction of F to the rectangle R_i gives a homotopy between f_i and g_i which is fully happening in either U or V. However, it is not path-homotopy since the end-points of these restrictions do not have to

match. Consider the paths which represent the way of these end points during the homotopy, i.e., define $\beta_i(t) = F(s_i, t)$. This way β_i is the path in X from $f(s_i)$ to $g(s_i)$ with both β_0 and β_n being constant paths at x and y, respectively. We would like to show that for all i,

$$f_i * \beta_i \simeq_p \beta_{i-1} * g_i.$$

For this consider the broken-line path along the bottom and right edges of R_i from $s_{i-1} \times 0$ to $s_i \times 1$, as shown on Figure 2.3.1. The composition of F with this path is equal to the path

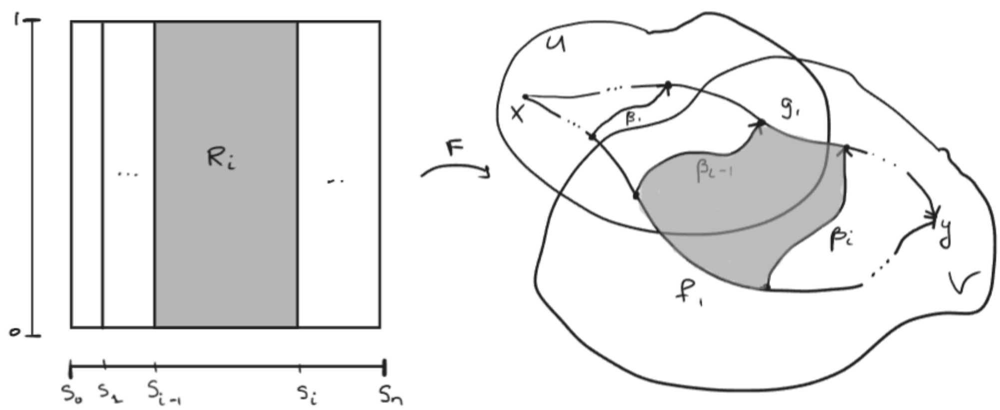


Figure 2.3.1

$f_i * \beta_i$. A similar thing happens when we take the broken-line path along the left and top edges of R_i and follow it by F - we obtain the path $\beta_{i-1} * g_i$. Since R_i is convex, by Example 1.1.6 there is a path homotopy in R_i between two broken-line paths and by Lemma 1.1.9 if we follow by F, we obtain a path homotopy between $f_i * \beta_i$ and $\beta_{i-1} * g_i$ which takes place in either U or V. Using the conditions (1) and (2) for σ, we get that

$$\sigma(f_i) *_h \sigma(\beta_i) = \sigma(\beta_{i-1}) *_H \sigma(g_i)$$

and, thus, we have

$$\sigma(f_i) = \sigma(\beta_{i-1}) *_H \sigma(g_i) *_H \sigma(\beta_i)^{-1}.$$

Similarly, since β_0 and β_n are constant maps, and identity elements get mapped to the identity elements, we have $\sigma(\beta_0) = \sigma(\beta_n) = e_H$. Now, we can compute using the definition

$$\tau(f) = \sigma(f_1) *_H \sigma(f_2) *_H \ldots *_H \sigma(f_n)$$

$$= \sigma(\beta_0) *_H \sigma(g_1) *_H \ldots *_H \sigma(g_n) *_H \sigma(\beta_n)^{-1}$$

$$= \sigma(g_1) *_H \ldots *_H \sigma(g_n) = \tau(g).$$

Therefore, we can deduce that $\tau(f_{j-1}) = \tau(f)$ for each j and so $\tau(f) = \tau(g)$.
Finally, let's show that τ satisfies the condition (2). Suppose we have a composition of paths

$f*g$ in X. Choose a subdivision $s_0 < ... < s_n$ of $[0,1]$ containing the point $1/2$ as a subdivision point s_k such that $f*g$ carries each sub-interval into either U or V. For $i = 1, \ldots, k$, the increasing linear map of I to $[s_{i-1}, s_i]$ followed by $f*g$ is the same as the increasing linear map from I to $[2s_{i-1}, 2s_i]$ followed by f; let's call the latter f_i. Similarly, for $i = k+1, \ldots, n$, the linear map of I to $[s_{i-1}, s_i]$ followed by $f*g$ is the same as the increasing linear map of I to $[2s_{i-1}-1, 2s_i-1]$ followed by g - let's call this map g_{i-k}. Using the subdivision $s_0, \ldots, s_n$ of $f*g$ from before, we have

$$\tau(f*g) = \sigma(f_1) *_H \ldots *_H \sigma(f_k) *_H \sigma(g_1) *_H \ldots *_H \sigma(g_{n-k}).$$

Using the subdivision $2s_0, \ldots, 2s_k$ of the path f we have

$$\tau(f) = \sigma(f_1) *_H \ldots *_H \sigma(f_k).$$

Similarly, using the subdivision $2s_k - 1, \ldots, 2s_n - 1$ of the path g we have

$$\tau(g) = \sigma(g_1) *_H \ldots *_H \sigma(g_{n-k}).$$

Therefore, (2) clearly holds since $\tau(f*g) = \tau(f) *_H \tau(g)$.
Step 4. For each loop f in X based at x_0 define $\Phi([f]) = \tau(f)$. The conditions (1) and (2) from above show that Φ is a well-defined homomorphism. To show that $\Phi \circ j_1 = \phi_1$ consider a loop f in U. Then

$$\Phi(j_1([f]_U)) = \Phi([f]) = \tau(f) = \rho(f) = \phi_1([f]_U).$$

Similarly, for a loop g in V we have

$$\Phi(j_2([g]_V)) = \Phi([g]) = \tau(g) = \rho(g) = \phi_2([g]_V).$$

□

The classical version of the same theorem assumes the modern version.

Theorem 2.3.2 (Seifert-van Kampen Theorem, classical version)**.** *Assume the hypotheses of the modern version of the theorem. Let $x_0 \in U \cap V$. Consider $j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the homomorphism of the free product that extends the homomorphisms j_1 and j_2. Then, j is surjective, and its kernel is the least normal subgroup N of $\pi_1(U, x_0) * \pi_1(V, x_0)$ that contains all elements represented by words of the form $i_1(g)^{-1} i_2(g)$ for $g \in \pi_1(U \cap V, x_0)$.*

The least normal subgroup of the noted product can also be described as a group generated by all elements of the form $i_1(g)^{-1} i_2(g)$ for $g \in \pi_1(U \cap V, x_0)$ and their conjugates.

Proof. Note that by Theorem 1.6.1, $\pi_1(X, x_0)$ is generated by the images of j_1 and j_2 and, thus, j is surjective. For the second part of the theorem, we will firstly show that $N \subseteq ker(j)$. Recall that the kernel of j is a normal subgroup of $\pi_1(U, x_0) * \pi_1(V, x_0)$. Note that it suffices to show that $i_1(g)^{-1} i_2(g)$ belongs to the kernel for all $g \in \pi_1(U \cap V, x_0)$. To the contrary, if there was an element of N which does not belong to the kernel, it would still belong to all normal subgroups of $\pi_1(U, x_0) * \pi_1(V, x_0)$, one of which is the kernel itself. Take an inclusion mapping $i : U \cap V \to X$, then

$$ji_1(g) = j_1 i_1(g) = i_*(g) = j_2 i_2(g) = ji_2(g),$$

which implies that $ji_1(g) = ji_2(g)$ and so $i_1^{-1}(g)i_2(g)$ is getting mapped to the identity element. Thus, $i_1^{-1}(g)i_2(g)$ belongs to the kernel of j. Moreover, j induces an epimorphism

$$k : \pi_1(U, x_0) * \pi_1(V, x_0)/N \to \pi_1(X, x_0)$$

since it is a composition of a homomorphism and an epimorphism. To show that N equals $ker(j)$, we need to show that k is injective since N is trivial. For that, it is enough to show that k has a left inverse.

Let H denote the group $\pi_1(U, x_0) * \pi_1(V, x_0)/N$. Also, let $\phi_1 : \pi_1(U, x_0) \to H$ be the inclusion map from $\pi_1(U, x_0)$ to the free product followed by the projection of the free product onto its quotient by N. Let $\phi_2 : \pi_1(V, x_0) \to H$ be defined similarly. Consider the diagram

$$\begin{tikzcd}
& \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X, x_0) \arrow[r, shift left, "\Phi"] & H \arrow[l, shift left, "k"] \\
& \pi_1(V, x_0) \arrow[u, "j_2"] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

Note that from the diagram we can see that $\phi_1 \circ i_1 = \phi_2 \circ i_2$. Moreover, if $g \in \pi_1(U \cap V, x_0)$, then $\phi_1(i_1(g))$ is the coset $i_1(g)N$ in H, and $\phi_2(i_2(G))$ is the coset $i_2(g)N$. Since $i_1(g)^{-1}i_2(g) \in N$, these two cosets are actually equal.

From the modern version of the Seifert-van Kampen Theorem we know that there exists a homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \circ j_1 = \phi_1$ and $\Phi \circ j_2 = \phi_2$. Let's show that Φ is the left inverse for k. For this to be true we need $\Phi \circ k$ to act as an identity on any generator of H, i.e., on any coset of the form gN, where $g \in \pi_1(U, x_0)$ or $g \in \pi_1(V, x_0)$. Suppose $g \in \pi_1(U, x_0)$, then we have

$$k(gN) = j(g) = j_1(g),$$

and so it follows that

$$\Phi(k(gN)) = \Phi(j_1(g)) = \phi_1(g) = gN,$$

which is exactly what we need. Similarly, one can show the same thing if $g \in \pi_1(V, x_0)$. $\square$

With Seifert-van Kampen's Theorem, we can get an exact formula for the fundamental group of a space X if we know the fundamental groups of a decomposition of X into U, V, and their intersection $U \cap V$. This theorem often is used when "gluing" familiar spaces together along a common and familiar subspace since instead of U and V we can take the covering $\{U_\alpha | \alpha \in \mathrm{A}\}$ of X by path-connected open sets such that the family is closed under finite intersection and all of its elements include the common point x_0.

Assuming the hypotheses of the Seifert-Van Kampen Theorem.

Corollary 2.3.3. If $U \cap V$ is simply-connected, then there is an isomorphism

$$k : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0).$$

Corollary 2.3.4. If V is simply-connected, then there is an isomorphism

$$k : \pi_1(U, x_0)/N \to \pi_1(X, x_0),$$

where N is the least normal subgroup of $\pi_1(U, x_0)$ containing the image of the homomorphism $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$

Definition 2.3.5. The **real projective plane** $\mathbb{RP}^2$ is a quotient space obtained from S^1 by identifying each point x with its antipode $-x$.

Theorem 2.3.6. *The projective plane $\mathbb{RP}^2$ is a compact surface with fundamental group isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Proof. To show that $\mathbb{RP}^2$ is a compact surface, we need to show that it is compact, Hausdorff, locally Euclidean and second countable. It is second countable since if S^2 has a countable basis $\{U_n\}$, the space $\mathbb{RP}^2$ would have a countable basis $\{p(U_n)\}$, where $p : S^2 \to \mathbb{RP}^2$ is a quotient map. The image space is clearly Hausdorff and locally Euclidean. Moreover, the space $\mathbb{RP}^2$ is compact as an image of a compact space S^2 under a continuous map p.
To calculate the fundamental group, consider $S^1 = U \cup V$ with $U = S^1 \backslash \{x\}$ for some point $x \notin S^1$ and V being an open neighbourhood around x. Then $U \cap V$ is an open disk around x. Note that the fundamental group of any open neighbourhood is 0, since it is a simply-connected space, so $\pi_1(V) = 0$. The open disk U is a deformation retraction of S^1, and, therefore, $\pi_1(U) \cong \mathbb{Z}$. Using Corollary 2.3.4, $\pi_1(\mathbb{RP}^2) \cong \langle a \rangle / \langle a | a^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$. □

2.4 The Fundamental Group of a Wedge of Circles

Definition 2.4.1. Consider a Hausdorff space $X = \bigcup_{i=1}^n S_i$, where each of S_i is homeomorphic to the unit circle S^1. If there is a point $p \in X$ such that $S_i \cap S_j = \{p\}$ whenever $i \neq j$, then we call the space X the **wedge (bouquet) of the circles** $\{S_i\}_{i=1}^n$.

Note that each space S_i is compact and, hence, closed in X. Moreover, since each S_i can be imbedded in the plane, the same can be said about the space X. In other words, if C_i denotes a circle of radius i in $\mathbb{R}^2$ with center at $(i, 0)$, then X is homeomorphic to $\bigcup_{i=1}^n C_i$.

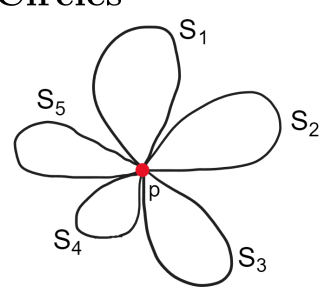


Figure 2.4.1: Example of the wedge of five circles.

Theorem 2.4.2. *Let X be the wedge of the circles $S_1, \ldots, S_n$ with the common point p. Then $\pi_1(X, p)$ is a free group. Moreover, if f_i is a loop in S_i that represents a generator of $\pi_1(S_i, p)$, then the loops $f_1, \ldots, f_n$ represent a system of free generators for $\pi_1(X, p)$.*

Let's prove a more general result than this for a space X which is a union of infinitely many circles, which all have a point in common.

Definition 2.4.3. Consider a space X, which is a union of the subspaces X_α for $\alpha \in J$. The topology of X is said to be **coherent** with the subspaces X_α provided a subset C of X is closed in X if $C \cap X_\alpha$ is closed in X_α for each α. An equivalent definition can be obtained with open sets instead.

In other words, a topological space is coherent with a family of subspaces if it is a topological union of those subspaces. In case of a finite collection of circles like before, X is the union of finitely many closed subspaces $\{X_i\}_{i=1}^n$ and so the topology of X is automatically coherent with these subspaces, since if $C \cap X_i$ is closed in X_i, it is also closed in X and, thus, C is the finite union of the sets $C \cap X_i$.

Definition 2.4.4. Let X be a space which is a union of the subspaces S_α, $\alpha \in J$, each of which is homeomorphic to S^1. If there is a point $p \in X$ such that for all $\alpha \neq \beta$ we have $S_\alpha \cap S_\beta = \{p\}$ and the topology of X is coherent with the subspaces S_α, then X is called the **wedge of the circles** S_α.

Note that the Hausdorff condition, which is included in Definition 2.4.1, is not included in the infinite case. But it is still required, it just follows from the coherent condition.

Lemma 2.4.5. Let X be the wedge of the circles $\{S_\alpha\}_{\alpha \in J}$. Then X is normal, and any compact subspace of X is contained in the union of finitely many circles S_α.

Proof. Firstly note that one-point sets are closed in X. Consider disjoint closed subsets A and B of the space X such that B does not contain the point p. Choose disjoint subsets U_α and V_α of S_α that are open in S_α and contain $\{p\} \cup (A \cap S_\alpha)$ and $B \cap S_\alpha$, respectively. Let $U = \bigcup_{\alpha \in J} U_\alpha$ and $V = \bigcup_{\alpha \in J} V_\alpha$. Then U and V are disjoint. Now, since all sets U_α contain p we have that $U \cap S_\alpha = U_\alpha$. Similarly, since none of the sets V_α contain p, we have $V \cap S_\alpha = V_\alpha$. Hence, U and V are open in X, and, thus, X is normal.

Consider a compact subspace C of X. Choose a point $x_\alpha \in C \cap (S_\alpha \backslash \{p\})$ if $C \cap (S_\alpha \backslash \{p\})$ is not empty. The set $D = \{x_\alpha\}$ is closed in X, since its intersection with each space S_α is either empty or a one-point set, which is closed in a Hausdorff space X. For the same reason, each subset of D is closed in X. Thus, D is a closed discrete subspace of X contained in C and since C is limit point compact, D must be finite. □

Theorem 2.4.6. *Let X be the wedge of circles $\{S_\alpha\}_{\alpha \in J}$ with the common point p. Then $\pi_1(X, p)$ is a free group. Moreover, if f_α is a loop in S_α representing a generator of $\pi_1(S_\alpha, p)$, then the loops $\{f_\alpha\}$ represent a system of free generators for $\pi_1(X, p)$.*

Proof. Let $i_\alpha : \pi_1(S_\alpha, p) \to \pi_1(X, p)$ be the homomorphism induced by inclusion and let G_α be the image of i_α. Note that if f is any loop in X based at p, then the image set of f is compact and so f lies in some finite union of subspaces S_α. Moreover, if f and g are two path-homotopic loops in X, then they are path-homotopic in some finite union of the subspaces of S_α by the preceding lemma.

To see that the groups $\{G_\alpha\}$ generate $\pi_1(X,p)$, consider a loop f in X. It must lie in $S_{\alpha_1} \cup \ldots \cup S_{\alpha_n}$ for some finite set of indices. By Theorem 2.4.2, we have $[f]$ as a product of elements of the groups $G_{\alpha_1}, \ldots, G_{\alpha_n}$. It follows that i_β is a monomorphism. In the case f is nulhomotopic in X, f must be path homotopic to a constant in some finite union of spaces S_α, so by Theorem 2.4.2, f is path homotopic to a constant in S_β.

Suppose there exists a reduced nonempty word $w = (g_{\alpha_1} \ldots g_{\alpha_n})$ in the elements of the groups G_α which represents the identity element of $\pi_1(X,p)$. Let f be a loop in X whose path-homotopy class is represented by w. Then f is path homotopic to a constant in X and so it is path homotopic to a constant in some finite union of subspaces S_α, which is not possible according to Theorem 2.4.2. □

Definition 2.4.7. Given two topological spaces X and Y with points $x_0 \in X$ and $y_0 \in Y$, the **wedge** $X \vee Y$ of X and Y is defined as the quotient space of their disjoint union where two copies of the base points (one in X and one in Y) are identified.

Example 2.4.8. Consider the wedge X of the spaces $X_1, \ldots, X_n$. Let's show that if for each i, the common point p is a deformation retract of an open set W_i of X_i, then $\pi_1(X,p)$ is the free product of the groups $\pi_1(X_i,p)$ relative to the monomorphisms induced by inclusion.
Consider the problem for the case when $X = X_1 \vee X_2$. We can assume that both X_1 and X_2 are path-connected since if C_i are the path components containing p in X_i, then $\pi_1(C_i,p) = \pi_1(X_i,p)$. Let $U = X_1 \cup W_2$ and let $V = X_2 \cup W_1$. Then both U and V are path-connected since their deformation retracts are X_1 and X_2, respectively, and $U \cap V = W_1 \cup W_2$ is simply-connected since its deformation retract is just the point p. Therefore, by Theorem 2.3.3, there is an isomorphism $\pi_1(X_1,p) * \pi_1(X_2,p) \cong \pi_1(X,p)$.

2.5 Adjoining a Two-Cell

Theorem 2.5.1. *Let X be a Hausdorff space and let A be a closed path-connected subspace of X. Suppose there is a continuous map $h : B^2 \to X$ which maps $Int(B^2)$ bijectively onto $X \backslash A$ and maps $S^1 = Bd(B^2)$ onto A. Let $p \in S^1$, $a = h(p)$ and let $k : (S^1,p) \to (A,a)$ be the restriction of h. Then the homomorphism*

$$i_* : \pi_1(A,a) \to \pi_1(X,a)$$

induced by the inclusion is surjective, and its kernel is the least normal subgroup of $\pi_1(A,a)$ containing the image of $k_* : \pi_1(S^1,p) \to \pi_1(A,a)$.

Proof. Step 1. Consider the origin 0 of B^2, its image $x_0 = h(0)$ in X and an open set $U = X\backslash\{x_0\}$ of X. Let's show that A is the deformation retract of U.

Let $C = h(B^2)$ and let $\pi : B^2 \to C$ be the restriction of h. Consider the map

$$\pi \times \mathbb{I} : B^2 \times I \to C \times I.$$

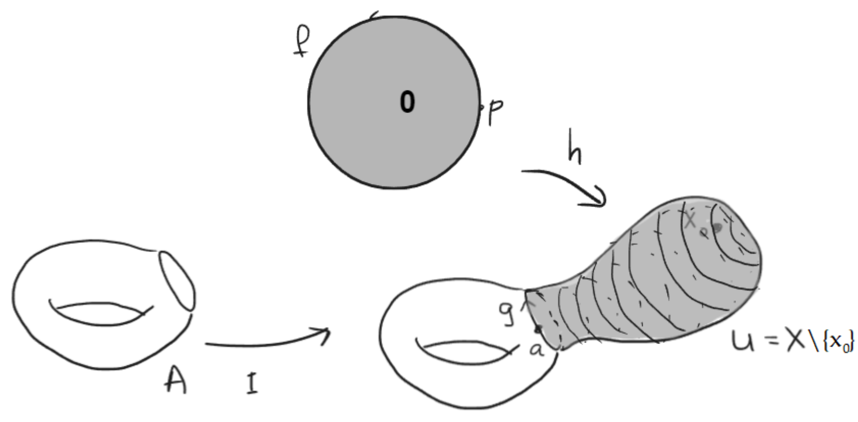


Figure 2.5.1: Representation of the construction discussed in Step 1.

Since $B^2 \times I$ is compact and $C \times I$ is Hausdorff, the map $\pi \times \mathbb{I}$ is closed. Since it is closed and surjective, it is a quotient map by definition. Its restriction

$$\pi' : (B^2 \backslash \{0\}) \times I \to (C \backslash \{x_0\}) \times I$$

is a quotient map as well, since its domain is open in $B^2 \times I$ and is saturated with respect to $\pi \times \mathbb{I}$. It is known that there is a deformation retraction of $B^2 \backslash \{0\}$ onto S^1, and so using the quotient map π' it can induce a deformation retraction of $C \backslash \{x_0\}$ to $\pi(S^1)$. We extend this deformation retraction to all $U \times I$ by letting each point of A remain fixed during the deformation. Therefore, A is a deformation retract of U.

Then by Theorem 1.4.6 the inclusion of A into U induces an isomorphism of fundamental groups and what we need to prove can be reduced to the following:

Let f be a loop whose class generates $\pi_1(S^1, p)$. Then the inclusion of U into X induces an epimorphism $\pi_1(U, a) \to \pi_1(X, a)$ whose kernel is the least normal subgroup containing the class of the loop $g = h \circ f$.

Step 2. In order to prove the reduced statement, consider the homomorphism $\pi_1(U, b) \to \pi_1(X, b)$ induced by inclusion relative to the base point b which does not belong to A.

Let b be any point of $U \backslash A$. Now, X is the union of the open sets U and $V = X \backslash A = \pi(Int(B^2))$. We know U is path-connected, since A is its deformation retraction. Because π is a quotient map, its restriction to $Int(B^2)$ is also a quotient map and hence homeomorphism. Thus, V is simply-connected. The set $U \cap V = V \backslash \{x_0\}$ is homeomorphic to $Int(B^2) \backslash \{0\}$, so it is path connected and its fundamental group is infinite cyclic. Since b is a point of $U \cap V$, by Theorem 2.3.4 the homomorphism $\pi_1(U, b) \to \pi_1(X, b)$ induced by the inclusion is surjective, and its kernel is the least normal subgroup containing the image of the infinite and cyclic group $\pi_1(U \cap V, b)$.

Step 3. Now, let's prove the result for a point a. Let q be the point of B^2 which is the midpoint of the line segment from 0 to p. Also, let $b = h(q)$ so b is a point in $U \cap V$. Let f_0 be a loop in $Int(B^2)\backslash\{0\}$ based at q that represents a generator of the fundamental group of this space. Then $g_0 = h \circ f_0$ is a loop in $U \cap V$ based at b that represents a generator of the fundamental group of $U \cap V$. By Step 2 we know that the homomorphism $\pi_1(U, b) \to \pi_1(X, b)$ induced by the inclusion is surjective and its kernel is the least normal subgroup containing the class of the loop $g_0 = h \circ f_0$.

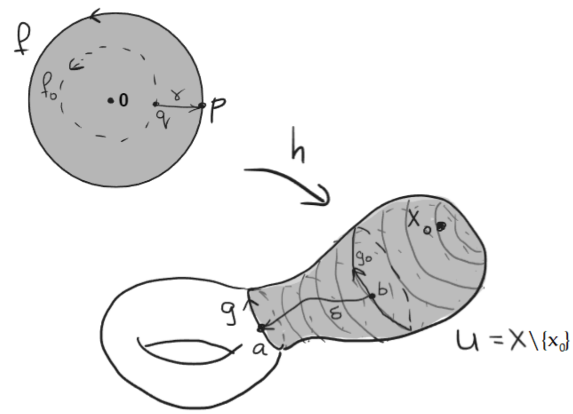


Figure 2.5.2: The situation described in Step 3.

To obtain the similar result for the point a, consider γ as the straight line path in B^2 from q to p and a path $\delta = h \circ \gamma$ in U from b to a. The isomorphism induced by the path δ commute with the homomorphisms, both denoted δ_0, induced by inclusion in the diagram:

$$\begin{array}{ccc} \pi_1(U, b) & \longrightarrow & \pi_1(X, b) \\ \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta_0} \\ \pi_1(U, a) & \longrightarrow & \pi_1(X, a) \end{array}.$$

Therefore, the homomorphism of $\pi_1(U, a)$ into $\pi_1(X, a)$ induced by inclusion is surjective and its kernel is the least normal subgroup containing the element $\delta_0([g_0])$.

The loop f_0 represents a generator of the fundamental group of $Int(B^2)\backslash\{0\}$ based at q. Then the loop $\overline{\gamma} * (f_0 * \gamma)$ represents a generator of the fundamental group of $B^2\backslash\{0\}$ based at p. Therefore, it is path homotopic to either f or its inverse. Suppose the latter: following the path homotopy by the map h, we note that $\overline{\delta} * (g_0 * \delta) \simeq_p g$ in U. Then $\hat{\delta}([g_0]) = [g]$ and the theorem follows. $\square$

Note that the unit ball in the Theorem above can be replaced with any space B which is homeomorphic to B^2. We call such space a **2-cell**. Then the space X in the Theorem is obtained by "adjoining" a 2-cell to A. In other words, the theorem above states that the fundamental group of X is obtained from the fundamental group of A by killing off the class $k_*[f]$, where $[f]$ generates $\pi_1(S^1, p)$.

Chapter 3

Classification of Surfaces

By now, we have built all the skills we need to be able to classify all the compact surfaces up to homeomorphism. This problem is more or less trivial for smaller dimensions, i.e., 0 and 1. For the smallest dimension, 0-dimensional connected manifold is just a point, which means that any 0-dimensional disconnected manifold is just a discrete set. In the case of 1 dimension, we would have a manifold homeomorphic to either a circle or a closed interval in case of compactness, otherwise, it has to be homeomorphic to the real line $\mathbb{R}$. The reader is welcome to read more about the one-dimensional case in [Dav87]. In this chapter, though, we would like to handle the case of compact two-dimensional manifolds.

3.1 Fundamental Groups of Surfaces

We would like to start with some construction. Let's look at surfaces which can be constructed as quotient spaces from a polygonal region in a plane.
Consider a point c of $\mathbb{R}^2$ and a number $a > 0$. Construct a circle in $\mathbb{R}^2$ with the center at c and with the radius a. Given a finite sequence $\theta_0 < \theta_1 < \ldots < \theta_n$ of real numbers, where $n \geqslant 3$ and $\theta_n = \theta_0 + 2\pi$, consider the points $p_i = c + a(\cos\theta_i, \sin\theta_i)$, which all lie on the circle described. They also are numbered in counterclockwise order around the circle with $p_n = p_0$. The line through p_{i-1} and p_i splits the circle and, as a result, the plane into two closed pieces. Let H_i be the one that contains all the points $\{p_i\}_{i=1}^n$, which we call **vertices**. Then the space $P = \bigcap_{i=1}^n H_i$ is what we call the **polygonal region** determined by the points $\{p_i\}_{i=1}^n$. The line segments $p_i p_{i+1}$ for all $i = 0, 1, \ldots, n-1$ with $p_n = p_0$ are called the **edges** of P. The union of all edges is what we call the **boundary** of P and, thus, the region $P \backslash Bd(P) = Int(P)$ is the **interior**.

Given a line segment L of $\mathbb{R}^2$, an **orientation** of L is the ordering of its end points: **initial point** a and **final point** b. In this case, we say that L is oriented from a to b. If L' is another line segment, oriented from c to d, then the order-preserving linear map of L onto L' is the homeomorphism h that carries the point $x = (1-s)a + sb$ of L to the point $h(x) = (1-s)c + sd$ of L'. Note that h is the straight-line homotopy between straight paths.

Using this, if two polygonal regions P and Q have the same number of vertices, $p_0, ..., p_n$ and $q_0, ..., q_n$, respectively, with $p_0 = p_n$ and $q_0 = q_n$, then combining all the separate homeomorphisms using the pasting lemma, we get a homeomorphism h of $Bd(P)$ with $Bd(Q)$ that carries the line segment from p_{i-1} to p_i by a linear map onto the line segment from q_{i-1} to q_i as shown on Figure 3.1.1. If p and q are fixed points of $Int(P)$ and $Int(Q)$, respectively, then this homeomorphism may be extended to a homeomorphism of P with Q which linearly maps a perpendicular from p to a point $x \in Bd(P)$ to a perpendicular from q to $h(x)$.

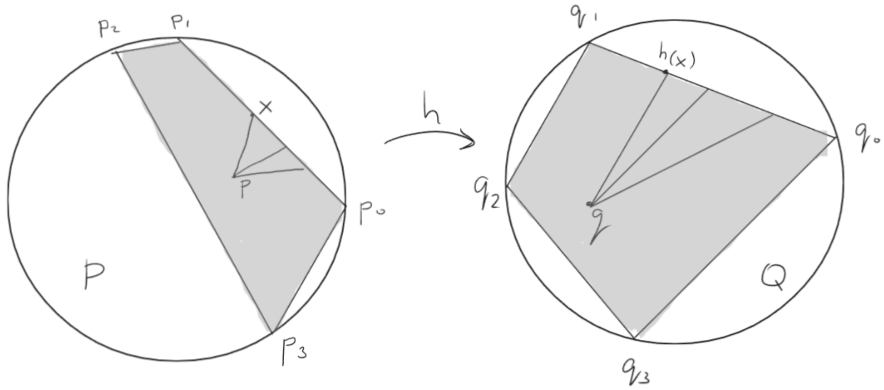


Figure 3.1.1

Definition 3.1.1. A **labelling** of the edges of a polygonal region P in the plane is a map from the set of edges of P to a set S called the **set of labels**. Given an orientation of each edge of P, and given a labelling of the edges of P, define an equivalence relation on the points of P as follows:

$$x \sim \begin{cases} x, & \text{if } x \in Int(P), \\ h(x), & \text{if } x \in Bd(P) \text{ and both } x \text{ and } h(x) \text{ belong to edges with the same label.} \end{cases}$$

The quotient space X obtained from this equivalence relation is said to have been obtained by **pasting the edges** of P together according to the given orientations and labelling.

Definition 3.1.2. Let $n \in \mathbb{N}\backslash\{1\}$ and let $r : S^1 \to S^1$ be rotation through the angle $2\pi/n$. Form a quotient space X from the unit ball B^2 by identifying each point x of S^1 with the points $r(x), r^2(x), \ldots, r^{n-1}(x)$. In this case, X is called the n**-fold dunce cap** and we will denote it as D_n.

Example 3.1.3. Consider the orientations and labelling of the edges of the triangular region pictured in Figure 3.1. All different orientations and labellings can give us different quotient spaces. Note that the provided labellings do not give a full list of all possible labellings for a triangle.

However, we would like to describe a method for specifying orientations and labels for the edges of a polygonal region without drawing a picture.

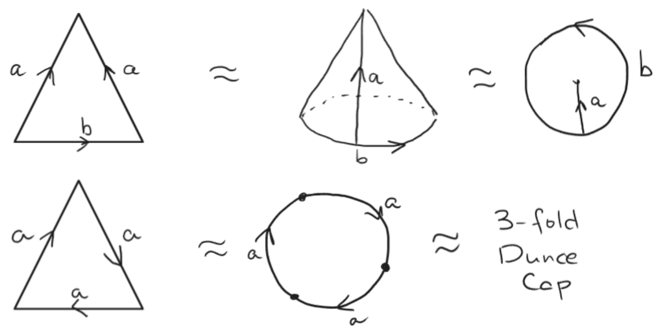


Figure 3.1: Different labellings for a triangle.

Definition 3.1.4. Let P be a polygonal region with vertices $p_0, ..., p_n$, where $p_0 = p_n$. Given orientations and a labelling of the edges of P, let $a_1, ..., a_m$ be the distinct labels that are assigned to the edges of P. For each k, let a_{i_k} be the label assigned to the edge $p_{k-1}p_k$, and let ϵ_k be equal $+1$ or -1 according to the orientation assigned to this edge, i.e., if it goes from p_{k-1} to p_k or the reverse. Then the number of edges of P, the orientations of the edges, and the labelling are completely specified by the symbol

$$w = (a_{i_1})^{\epsilon_1}(a_{i_2})^{\epsilon_2} \dots (a_{i_n})^{\epsilon_n},$$

which is called a **labelling scheme** for the edges of P.

We can omit the positive exponents in the labelling scheme to get the scheme to be looking like words which we have been working with in the previous chapter. Recall the first figure in Example 3.1.3: the labelling scheme there can be written as $a^{-1}ba$ if we take p_0 to be a top vertex of the triangle. If we decide to switch p_0 we would get the schemes baa^{-1} and $aa^{-1}b$. It is clear that a cyclic permutation of the terms of the labelling scheme will change the end space X formed by using the scheme only up to homeomorphism.

Example 3.1.5. A sphere can be constructed by pasting the edges of a square with the labelling scheme $aa^{-1}bb^{-1}$. Torus T_2 can also be constructed by pasting the edges of a square, but with the labelling scheme $aba^{-1}b^{-1}$ as shown on Figure 3.1.2.

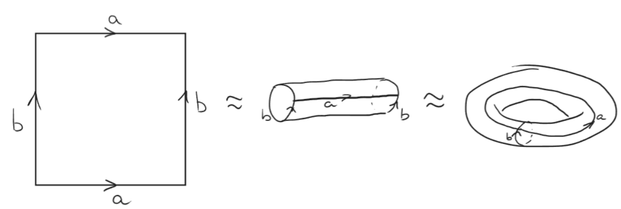


Figure 3.1.2: Construction of a torus using a square labelling.

Example 3.1.6. Recall that we defined the projective plane to be homeomorphic to the quotient space of the unit ball B^2 obtained by identifying every point of the boundary with its antipode. Since the unit square is homeomorphic to the unit ball, we can specify the same space by the labelling scheme *abab*.

Theorem 3.1.7. *Let X be the space obtained from a finite collection of polygonal regions by pasting edges together according to some labelling scheme. Then X is a compact Hausdorff space.*

Proof. We will prove the case where X is obtained from a single polygonal region. This can be extended to an arbitrary collection of polygonal regions. Firstly, note that since the image of a compact space under a continuous map is compact and the quotient map is continuous, X is compact. To show that X is also Hausdorff, let's use the Lemma A.0.27 and instead show that the quotient map π is a closed map. For this, we need to show that for each closed set C of P, the set $\pi^{-1}(\pi(C))$ is closed in P. Now, the set $\pi^{-1}(\pi(C))$ consists of all points of C and all points of P which are pasted to points of C by the map π. To determine these points consider the compact subspaces $C \cap e$ of P for each edge e. If e_i is an edge that gets pasted to e, and if $h_i : e_i \to e_i$ is the pasting homeomorphism, then the set $D_e = \pi^{-1}(\pi(C)) \cap e$ contains the space $h_i(C_{e_i})$. Thus, D_e equals the union of C_e and the spaces $h_i(C_{e_i})$, as e_i ranges through all the edges of P which are pasted to e. Since this union is compact, it is closed in e and in P. Since $\pi^{-1}(\pi(C))$ is the union of the set C and sets D_e, as e ranges over all edges of P, it is closed in P, as needed. □

Note that if X is obtained by pasting the edges of a polygonal region together, the quotient map π may map all the vertices of the polygonal region to a single point of X, or it may not. In the case of the torus, the quotient map does satisfy this condition, while in the case of the sphere, it does not.

Theorem 3.1.8. *Let P be a polygonal region; let*

$$w = (a_{i_1})^{\epsilon_1}(a_{i_2})^{\epsilon_2}\dots(a_{i_n})^{\epsilon_n}$$

be a labelling scheme for the edges of P. Let X be the resulting quotient space and let $\pi : P \to X$ be the quotient map. If π maps all the vertices of P to a single point x_0 of X, and if $a_1, \dots, a_k$ are the distinct labels that appear in the labelling scheme, then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on k generators $\alpha_1, ..., \alpha_k$ by the least normal subgroup containing the element

$$(\alpha_{i_1})^{\epsilon_1}(\alpha_{i_2})^{\epsilon_2}\dots(\alpha_{i_n})^{\epsilon_n}.$$

Proof. The map π sends all vertices of P to a single point of X. Therefore, the space $A = \pi(Bd(P))$ is a wedge of k circles. For each i, choose an edge of P which is labelled a_i. Consider the linear map f_i of I onto the chosen edge oriented counterclockwise and let $g_i = \pi \circ f_i$. Then the loops $g_1, \dots, g_k$ represent the set of free generators for $\pi_1(A, x_0)$. The loop f going around $Bd(P)$ once in the clockwise direction generates the fundamental group of $Bd(P)$ and, thus, the loop $\pi \circ f$ equals the loop $(g_{i_1})^{\epsilon_1} \dots (g_{i_n})^{\epsilon_n}$. Now, the needed result follows from Theorem 2.5.1. □

Definition 3.1.9. Consider the space T_n obtained from a $4n$-sided polygonal region P with the labelling scheme

$$(a_1 b_1 a_1^{-1} b_1^{-1})(a_2 b_2 a_2^{-1} b_2^{-1}) \dots (a_n b_n a_n^{-1} b_n^{-1}).$$

This space is called the n**-fold connected sum of tori** or n**-torus**.

In other words, to construct the 2-fold torus, we can consider the polygonal region P. See Figure 3.1.3. If we split the polygonal region P along the indicated line c, each of the resulting pieces represents a torus with an open disc removed. Another way to construct such surface would be taking two copies of the torus T^2, deleting a small open disk from each of them, and pasting the remaining pieces together along their edges. A similar argument for both of construction techniques shows the construction of the 3-fold torus $T\#T\#T$ and so on. See Figure 3.1.4.

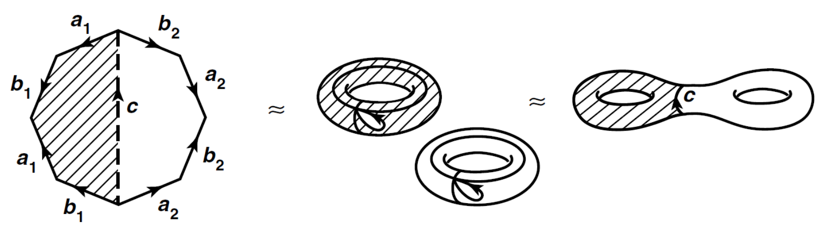


Figure 3.1.3: Construction of $T\#T$ from a polygonal region [Mun00].

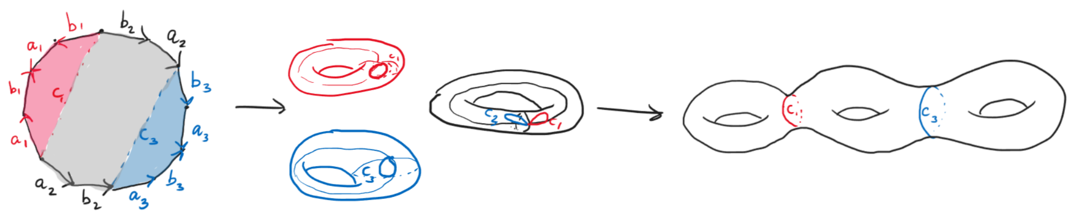


Figure 3.1.4: Construction of $T\#T\#T$ from a polygonal region.

Theorem 3.1.10. *Let X denote the n-fold torus. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on the $2n$ generators $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n$ by the least normal subgroup containing the element*

$$[\alpha_1, \beta_1][\alpha_2, \beta_2] \dots [\alpha_n, \beta_n],$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$.

Proof. To be able to use Theorem 3.1.8 we need to show that the quotient map sends all the vertices of the space X to a single point x_0 of X since all the labels are distinct by definition. Note that every n-fold torus can be split up into n separated tori, which means that it is

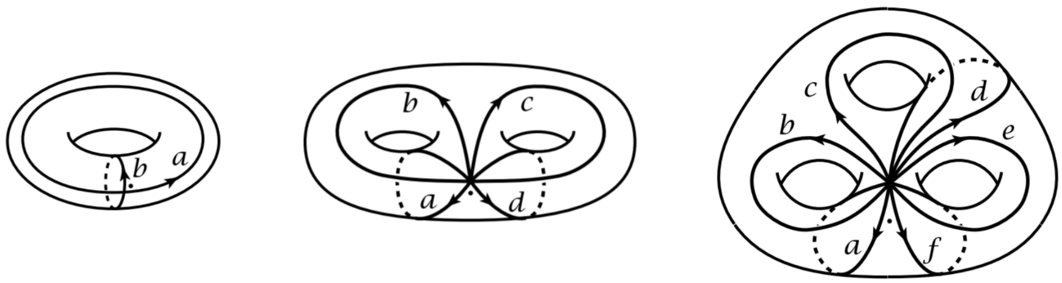


Figure 3.1.5: Generators for T_1, T_2 and T_3, respectively [Hat02].

enough to show that all vertices of a single torus get mapped to a single point. That is true by construction. For instance, consider a double torus with its generators, all represented in the middle of Figure 3.1.5. Loops a and b, which are generated from one of the tori, get combined with loops c and d from the construction of the second tori. All together these four generators can make up any loop on the double torus. □

Definition 3.1.11. Let $m > 1$. Consider the space obtained from a $2m$-sided polygonal region P in the plane by means of the labelling scheme

$$(a_1a_1)(a_2a_2)\dots(a_ma_m).$$

This space is called the m**-fold connected sum of projective planes**, or simply the m**-fold projective plane**, and denoted by $\mathbb{RP}_m = \mathbb{RP}^2\#\dots\#\mathbb{RP}^2$.

Similarly to Theorem 3.1.10, we get the following result for $\mathbb{RP}_m$.

Theorem 3.1.12. *Let X denote the m-fold projective plane. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on m generators $\alpha_1, \dots, \alpha_m$ by the least normal subgroup containing the element*

$$(\alpha_1)^2(\alpha_2)^2\dots(\alpha_m)^2.$$

Example 3.1.13. The **Klein bottle** K is the space obtained from a square by means of the labelling scheme $aba^{-1}b$ as shown on Figure 3.1.6. Moreover, by Theorem 3.1.8 we know the presentation of its fundamental group: $\pi_1(K) = \langle a, b | aba^{-1}b\rangle$.

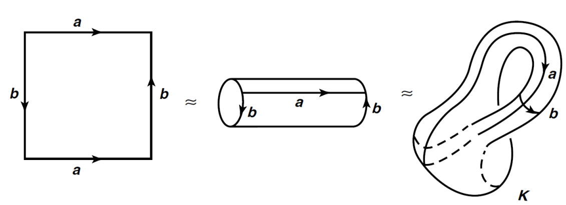


Figure 3.1.6: Construction of the Klein Bottle [Mun00].

3.2 Homology of Surfaces

By this point we know how to construct a surface or get a presentation of a surface, but we do not know yet how to compare its fundamental group to a fundamental group of another surface. This is what we are going to explore in this section.

Definition 3.2.1. Let X be a path-connected space with $x_0 \in X$. Define the **first homology group** of X as

$$H_1(X, x_0) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)].$$

We know that if X is a path-connected space, and if α is a path in X from x_0 to x_1, then there is an isomorphism $\tilde{\alpha}$ of $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$, but the isomorphism depends on the choice of the path α. We would like to verify a stronger result for the group $H_1(X)$. In this case, the isomorphism of the "abelianized fundamental group" based at x_0 with one based at x_1, induced by the path α, is independent of the choice of the path α.

To verify the independence, it suffices to show that if α and β are two paths from x_0 to x_1, then the path $\alpha * \overline{\beta}$ induced the identity isomorphism of $\pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$ with itself. Indeed, if $[f] \in \pi_1(X, x_0)$, then we have

$$\tilde{g}[f] = [\overline{g} * f * g] = [g]^{-1} * [f] * [g].$$

When we pass to the cosets in the abelian group $\pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$, we see that $\tilde{g}$ induces the identity map.

One can show that the base point is not relevant in the notation of this group similarly to the fundamental group of a path-connected space. This being said, we will denote it by $H_1(X)$ instead.

Showed independence of the base point and a path shows that to differentiate between two surfaces one can compute their homology groups instead of fundamental groups. To do this we need the following result.

Theorem 3.2.2. *Let F be a group with N being a normal subgroup of F. Consider the projection $q : F \to F/N$. Then the projection homomorphism $p : F \to F/[F, F]$ induces an isomorphism*

$$\phi : q(F)/[q(F), q(F)] \to p(F)/p(N).$$

In other words, if one divides F by N and abelianizes the quotient, they would obtain the same result as if we abelianize F first and then divide by the image of N in this abelianization.

Proof. Consider the projection homomorphisms p, q, r, s as given in the following diagram, where $q(F) = F/N$ and $p(F) = F/[F, F]$.

$$\begin{tikzcd}
 & q(F) \arrow[r, "s"] \arrow[ddr, "u"] & q(F)/[q(F), q(F)] \arrow[dd, shift right, "\phi"'] \\
F \arrow[ur, "q"] \arrow[dr, "p"'] & & \\
 & p(F) \arrow[r, "r"'] \arrow[uur, "v"] & p(F)/p(N) \arrow[uu, shift right, "\psi"']
\end{tikzcd}$$

Since $r \circ p$ maps N to the identity, it induces a homomorphism $u : q(F) \to p(F)/p(N)$. Now, since the image group is abelian, the homomorphism u induces a homomorphism $\phi : q(F)/[q(F), q(F)] \to p(F)/p(N)$. On the other hand, since $s \circ q$ maps F onto an abelian group, it also induces a homomorphism $v : p(F) \to q(F)/[q(F), q(F)]$. Because $s \circ q$ maps N to the identity, the same thing is done by $v \circ p$ and, thus, v induces a homomorphism $\psi : p(F)/p(N) \to q(F)/[q(F), q(F)]$.

We can describe the homomorphisms ϕ and ψ in a similar way such that they are inverses of each other. For instance, for a given y in $q(F)/[q(F), q(F)]$, choose an element x of F such that $s \circ q(x) = y$. Then $\phi(y) = r(p(x))$. □

Corollary 3.2.3. Let F be a free group with free generators $\alpha_1, \alpha_2, \ldots, \alpha_n$. Let N be the least normal subgroup of F containing the element x of F and let $G = F/N$. Consider the projection $p : F \to F/[F, F]$. Then $G/[G, G]$ is isomorphic to the quotient of $F/[F, F]$, which is free abelian with basis $p(\alpha_1), \ldots, p(\alpha_n)$, by the subgroup generated by $p(x)$.

Proof. The group N is generated by x and all of its conjugates. Also, the group $p(N)$ is generated by $p(x)$ since p is a projection. Therefore, by the preceding theorem, the corollary follows. □

Theorem 3.2.4. *If X is the n-fold connected sum of tori, then $H_1(X)$ is a free abelian group of rank $2n$.*

Proof. By Theorem 3.1.10, the fundamental group of the n-fold tori is isomorphic to the quotient of the free group on the $2n$ generators $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ by the least normal subgroup containing the element

$$[\alpha_1, \beta_1][\alpha_2, \beta_2] \ldots [\alpha_n, \beta_n],$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Now, using Corollary 3.2.3, $H_1(X)$ is isomorphic to the quotient of the free abelian group F' on the set of generators $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$ by the subgroup generated by the element $[\alpha_1, \beta_1] \ldots [\alpha_n, \beta_n]$. Since the group F' is abelian, the element equals to the identity element. Using presentations we can write the following:

$$H_1(T_n) = (\pi_1(T_n))_{ab} = \langle \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n | [\alpha_1, \beta_1] \ldots [\alpha_n, \beta_n] \rangle_{ab} = \mathbb{Z}^{\oplus 2n}.$$

□

Theorem 3.2.5. *If X is the m-fold connected sum of projective planes, then the torsion subgroup $T(X)$ of $H_1(X)$ has order 2, and $H_1(X)/T(X)$ is a free abelian group of rank $m - 1$.*

Proof. By Theorem 3.1.12, $\pi_1(X)$ is isomorphic to the quotient of the free group F' on the set of generators $\alpha_1, \ldots, \alpha_m$ by the subgroup generated by $(\alpha_1)^2 \ldots (\alpha_m)^2$. Let $\beta = \alpha_1 \cdot \ldots \cdot \alpha_m$. The torsion subgroup $T(X)$ is generated by β^2 and, thus, the order of it is 2. Moreover, the elements $\alpha_1, \ldots, \alpha_{m-1}, \beta$ form a basis for F'. Now, by Corollary 3.2.3, $H_1(X)$ is isomorphic to a quotient of a free abelian group generated by m elements by the subgroup generated by the image of β^2. Then $H_1(X)$ is isomorphic to the quotient of the m-fold Cartesian product

$\mathbb{Z} \times \ldots \times \mathbb{Z}$ by the subgroup $0 \times \ldots \times 0 \times 2\mathbb{Z}$, which is the same as $\mathbb{Z}^{m-1} \oplus \mathbb{Z}_2$. By Theorem 2.1.11, both $H_1(X)/T(X)$ and $T(X)$ are free abelian. Thus, $T(X)$ is isomorphic to $\mathbb{Z}_2$ and so $H_1(X)/T(X)$ is isomorphic to $\mathbb{Z}^{m-1}$. As a result, $H_1(X)/T(X)$ is a free abelian group of rank $m-1$, as required. □

Now, as we have computed the first homology groups for the connected sum of tori and the connected sum of projective planes, we get the following.

Corollary 3.2.6. Let T_n and $\mathbb{RP}_m$ denote the n-fold connected sum of tori and the m-fold connected sum of projective planes, respectively. The the surfaces $S^2, T_1, T_2, \ldots, \mathbb{RP}_1, \mathbb{RP}_2, \ldots$ are all topologically distinct.

3.3 Cutting and Pasting

Now, as we have developed some algebraic techniques, we also need to catch up on some geometric techniques for computing the fundamental group. These "cut-and-paste" techniques let us see how a space X can be represented by different collections of polygonal regions and different labelling schemes.

First, let's consider cutting. Let P be a polygonal region with successive vertices $p_0, p_1, \ldots, p_n = p_0$. Given k with $1 < k < n-1$, consider the polygonal regions Q_1 with successive vertices $p_0, p_1, \ldots, p_k, p_0$, and Q_2 with successive vertices $p_0, p_k, \ldots, p_n = p_0$. These regions have the edge p_0p_k in common and the region P is their union.

Now, let's move (by a translation in $\mathbb{R}^2$) one of the regions, for instance, Q_1, away from the region Q_2 to obtain two polygonal regions with empty intersection. Call this new region Q_1'. Then the regions Q_1' and Q_2 are said to have been obtained by **cutting P apart** along the line from p_0 to p_k. See Figure 3.3.1 from right to left. Note that the region P is homeomorphic to the quotient space of Q_1' and Q_2 obtained by pasting the edge of Q_1' going from q_0 to q_k to the edge of Q_2 going from p_0 to p_k, by a linear map of one edge onto the other. For the reverse operation, suppose we are given two disjoint polygonal regions Q_1' with

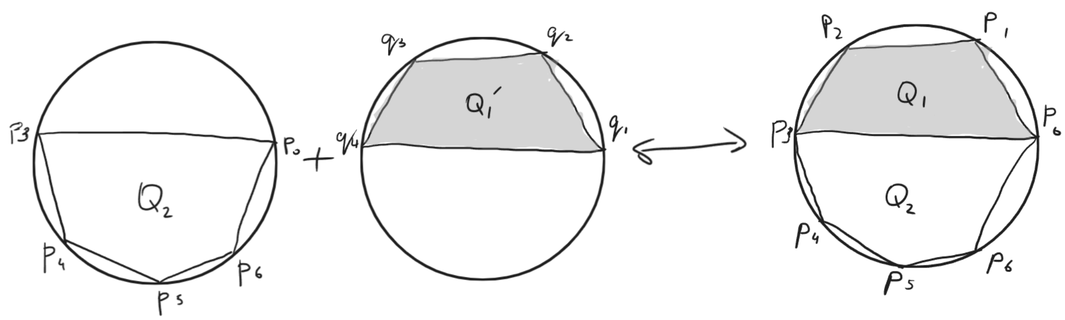


Figure 3.3.1: Visualization of the pasting operation for a polygonal region P with 7 vertices.

successive vertices $q_0, \ldots, q_k, q_0$, and Q_2, with successive vertices $p_0, p_k, \ldots, p_n = p_0$. Also, suppose we form a quotient space by pasting the edge of Q'_1 from q_0 to q_k onto the edge of Q_2 from p_0 to p_k, by a order-preserving linear map of one edge onto the other.

The points of Q_2 lie on a circle and are arranged in counterclockwise fashion. Let us choose points $p_1, \ldots, p_{k-1}$ on the same circle in such a way that $p_0, p_1, \ldots, p_{k-1}, p_k$ are arranged in counterclockwise order, and let Q_1 be the polygonal region with these as successive vertices. There is a homeomorphism of Q'_1 onto Q_1 that carries q_i to p_i for each i and maps the edge q_0q_k of Q'_1 linearly onto the edge p_0p_k of Q_2. Therefore, the quotient space before is homeomorphic to the region P which is the union of Q_1 and Q_2. We say that P is obtained by **pasting Q'_1 and Q_2 together** along the indicated edge. See Figure 3.3.1 from left to right. We can summarize this as a theorem.

Theorem 3.3.1. *Suppose X is the space obtained by pasting the edges of m polygonal regions together according to the scheme*

$$y_0y_1, w_2, \ldots, w_m.$$

Let c be a label not appearing anywhere in the scheme above. If both y_0 and y_1 have length at least two, then X can also be obtained by pasting the edges of $m+1$ polygonal regions according to the scheme

$$y_0c^{-1}, cy_1, w_2, \ldots, w_m.$$

Note that the converse of this statement also holds due to the nature of cutting.

We can make a list of **elementary scheme operations** which we are allowed to perform without affecting the resulting space X.

1. Cut: replacing the scheme $w_1 = y_0y_1$ with schemes y_0c^{-1} and cy_1, provided that c does nor appear elsewhere in the w_1 and both y_0 and y_1 have length at least two.

2. Paste: replacing the scheme y_0c^{-1} and cy_1 by the scheme y_0y_1, provided c does not appear elsewhere in the total scheme.

3. Relabel: replacing all occurrences of any given label by some other label which does not appear anywhere in the total scheme. Similarly, one can change the sign of the exponent of all occurrences of a label.

4. Permute: replacing one of the schemes w_i by a cyclic permutation of w_i. In other words, the scheme now begins with a different vertex without changing anything else.

5. Flip: replacing the scheme $(a_{i_1})^{\epsilon_1} \ldots (a_{i_n})^{\epsilon_n}$ with its formal inverse $(a_{i_n})^{-\epsilon_n} \ldots (a_{i_1})^{-\epsilon_1}$.

6. Cancel: deleting pairs aa^{-1} in the scheme $y_0aa^{-1}y_1$ given that a does not appear elsewhere in the total scheme and both y_0 and y_1 have the length at least two. To see the geometric meaning of this operation, consider Figure 3.3.2.

7. Uncancel: the reverse operation of the previous operation.

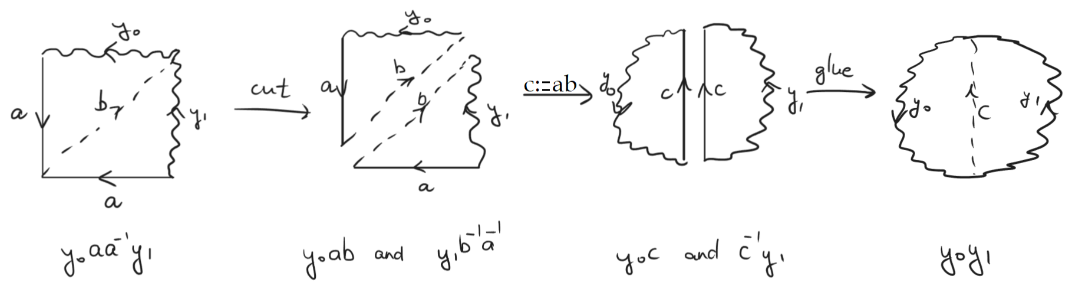


Figure 3.3.2: Visualization of the cancel operation.

Definition 3.3.2. Two labelling schemes for collections of polygonal regions are **equivalent** if one can be obtained from the other by a sequence of elementary scheme operations.

Note that since each elementary operation has its inverse operation also on the list of the elementary operations, the notion of equivalence, which was introduced above, is an equivalence relation.

Example 3.3.3. Going back to Example 3.1.13, we already know that the Klein bottle K is the space obtained from the labelling scheme $aba^{-1}b$. It is homeomorphic to the 2-fold projective plane $\mathbb{RP}^2 \# \mathbb{RP}^2$, which can be seen through the following elementary operations and visualized as shown on Figure 3.3.3.

$$
\begin{aligned}
aba^{-1}b &\sim abc^{-1} \text{ and } ca^{-1}b && \text{cutting} \\
&\sim c^{-1}ab \text{ and } b^{-1}ac^{-1} && \text{permuting and flipping} \\
&\sim c^{-1}aac^{-1} && \text{pasting} \\
&\sim aacc && \text{permuting and relabelling}
\end{aligned}
$$

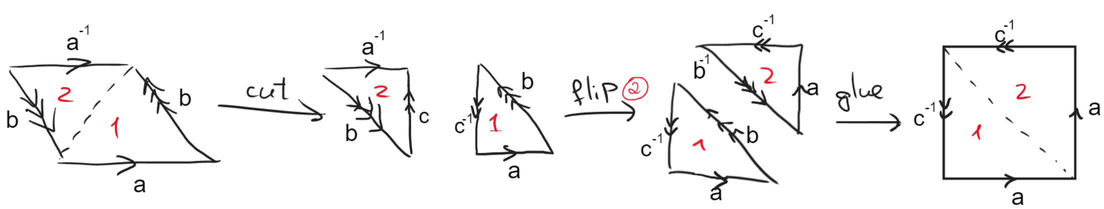


Figure 3.3.3: Klein Bottle transformed into $\mathbb{RP}^2 \# \mathbb{RP}^2$.

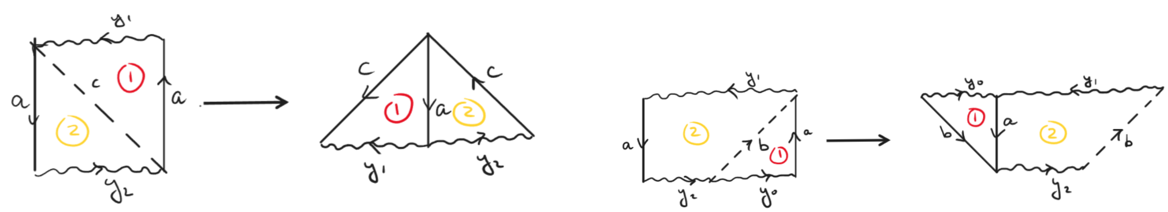
3.4 The Classification Theorem

First we would like to show that every space obtained by pasting the edges of the polygonal region together in pairs is homeomorphic either to S^2, T_n or $\mathbb{RP}_m$.
Consider polygonal regions $P_1, \dots, P_k$ with labelling schemes $w_1, \dots, w_k$. We call a scheme **proper** if each label appears exactly twice in a labelling scheme. Note that a proper scheme remains proper after any elementary operation from the previous section.

Definition 3.4.1. Let w be a proper labelling scheme for a single polygonal region. We call w of **torus type** if each label in it appears exactly twice, each time with a different exponent. Otherwise, we call w of **projective type**.

Consider a scheme w of projective type. This being said w has either a label not appearing twice or a label appearing twice but with the same exponent. Consider the latter case, i.e., $w = [y_0]a[y_1]a[y_2]$, where writing $[y_i]$ means that y_i may be empty.

Lemma 3.4.2. Consider a proper scheme $w = [y_0]a[y_1]a[y_2]$, where some of the y_i for $i = 0, 1, 2$ may be empty. Then $w \sim aa[y_0 y_1^{-1} y_2]$.



(a) Case where y_0 is empty.

(b) General case.

Figure 3.4.1: Labelling schemes operations following the proof of the Lemma 3.4.1.

Proof. Let's first assume that y_0 is empty. In this case we need to show that $a[y_1]a[y_2] \sim aa[y_1^{-1}][y_2]$. In case of y_1 being empty as well, we have an automatic equality. Otherwise, in case of y_2 being empty, one has $a[y_1]a$ being equal to $a^{-1}[y_1^{-1}]a^{-1}$, then $a^{-1}a^{-1}[y_1^{-1}]$, which is also equal to $aa[y_1^{-1}]$. Now, if neither of $[y_1]$ or $[y_2]$ is empty, one has to do a similar sequence of operations to show the same thing. Consider operations shown on Figure 3.4.1a:

$$
\begin{aligned}
a[y_1]a[y_2] &\sim a[y_1]c \text{ and } c^{-1}a[y_2] && \text{cutting}\\
&\sim a[y_1]c \text{ and } c[y_2^{-1}]a^{-1} && \text{permuting and flipping}\\
&\sim [y_1]cc[y_2^{-1}] && \text{pasting}\\
&\sim aa[y_1^{-1}][y_2] && \text{permuting and relabelling.}
\end{aligned}
$$

Now we can consider the general case with y_0 not being empty. In case of both y_1 and y_2 being empty, one can permute the scheme and get the wanted result. Otherwise, consider the sequence of operations shown on Figure 3.4.1b:

$$
\begin{aligned}
[y_0]a[y_1]a[y_2] &\sim [y_0]ab^{-1} \text{ and } b[y_1]a[y_2] && \text{cutting}\\
&\sim b^{-1}[y_0]a \text{ and } [y_1^{-1}]b^{-1}[y_2^{-1}]a^{-1} && \text{permuting and flipping}\\
&\sim b^{-1}[y_0][y_1^{-1}]b^{-1}[y_2^{-1}] && \text{pasting}\\
&\sim b[y_2]b[y_1y_0^{-1}] && \text{permuting and relabelling}\\
&\sim bb[y_2^{-1}y_1y_0^{-1}] && \text{by the case prior when } y_0 \text{ is empty}\\
&\sim [y_0y_1^{-1}y_2]b^{-1}b^{-1} && \text{flipping}\\
&\sim aa[y_0y_1^{-1}y_2] && \text{permuting and relabelling.}
\end{aligned}
$$

□

Corollary 3.4.3. If w is a scheme of projective type, then it is equivalent to a scheme of the same length and of the form $(a_1a_1)(a_2a_2)\dots(a_ka_k)w_1$, where $k \geqslant 1$ and w_1 is either of torus type or empty.

Proof. The scheme w can be written as $[y_0]a[y_1]a[y_2]$, which by Lemma 3.4.2 is equivalent to $w' = aaw_1$ with the same length as w. If w_1 is of torus type, we are done. Otherwise, rewrite w' as $aa[z_0]b[z_1]b[z_2] = [aaz_0]b[z_1]b[z_2]$. By the same Lemma, this scheme is equivalent to $w'' = bb[aaz_0z_1^{-1}z_2] = bbaaw_2$ with the same length as w. If w_2 is of torus type, we are done. Otherwise, the argument can be continued until we reach the torus type scheme. □

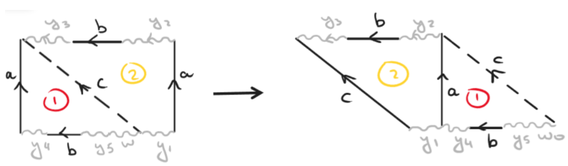
Lemma 3.4.4. Consider a proper scheme $w = w_0w_1$, where w_1 is a scheme of torus type which does not contain two adjacent terms having the same label. Then w is equivalent to a scheme $w' = w_0w_2$, where w_2 has the same length as w_1 and has the form $w_2 = aba^{-1}b^{-1}w_3$, where w_3 is either of torus type or empty.

Proof. Step 1. First, let's show that w can be written in the form

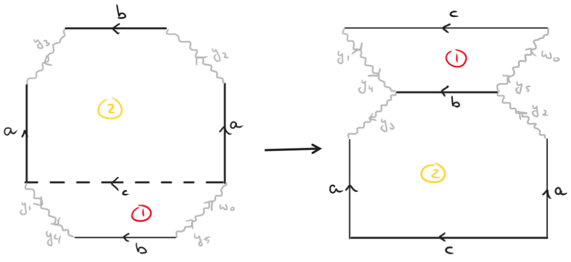
$$w = w_0[y_1]a[y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5], \tag{3.4.1}$$

where some of the y_i might be empty. Let a be the label whose occurrences with opposite exponents are the close together as possible. Since these occurrences are non-adjacent, there is at least one other label in between - call this label b. We can, without loss of generality, assume that b and a appear with positive exponent first. Otherwise, we just need to switch the labels. Now, since a and a^{-1} are the closest to each other such labels, the label b^{-1} cannot appear before a^{-1}. Thus, it has to come after a^{-1} or before a. In the first case, we are finished. The second scheme is the same if one switch the label a to b^{-1} and b to a.
Step 2. Consider the form 3.4.1 of w and rewrite it as

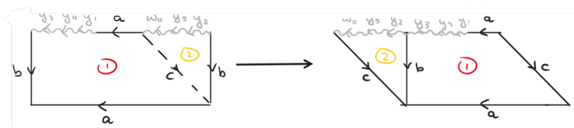
$$w = w_0[y_1]a[y_2by_3]a^{-1}[y_4b^{-1}y_5].$$



(a) Step 2 operation.



(b) Step 3, second case.



(c) Step 4, second case.

Figure 3.4.2: Labelling schemes operations following the proof of the Lemma 3.4.2.

Now, consider the cutting and pasting operation represented in Figure 3.4.2a. We have the following result.

$$\begin{aligned} w \sim & w_0 c[y_2 b y_3] c^{-1}[y_1 y_4 b^{-1} y_5] \\ \sim & w_0 a[y_2] b[y_3] a^{-1}[y_1 y_4] b^{-1}[y_5] = w' && \text{relabelling.} \end{aligned}$$

Step 3. If all the schemes y_1, y_4, y_5 and w_0 are empty, then one gets

$$\begin{aligned} w' &= a[y_2] b[y_3] a^{-1} b^{-1} \\ &\sim b[y_3] a^{-1} b^{-1} a[y_2] && \text{permuting} \\ &\sim a[y_3] b a^{-1} b^{-1}[y_2] = w'' && \text{relabelling.} \end{aligned}$$

Otherwise, we can apply the operations represented in

$$\begin{aligned} w' &= w_0 a[y_2] b[y_3] a^{-1}[y_1 y_4] b^{-1}[y_5] \\ &\sim w_0 c[y_1 y_4 y_3] a^{-1} c^{-1} a[y_2 y_5] && \text{Figure 3.4.2b} \\ &\sim w_0 a[y_1 y_4 y_3] b a^{-1} b^{-1}[y_2 y_5] && \text{relabelling.} \end{aligned}$$

Both times end scheme can be put as $w'' = w_0 a[y_1 y_4 y_3] b a^{-1} b^{-1}[y_2 y_5]$.
Step 4. Similar to the previous step, if the schemes w_0, y_1 and y_2 are empty, one gets

$$\begin{aligned} w' &= a[y_1 y_4 y_3] b a^{-1} b^{-1} \\ &\sim b a^{-1} b^{-1} a[y_1 y_4 y_3] && \text{permuting} \\ &\sim a b a^{-1} b^{-1}[y_1 y_4 y_3] = w''' && \text{relabelling.} \end{aligned}$$

Otherwise, we can apply the operations

$$\begin{aligned} w'' &= w_0 a[y_1y_4y_3]ba^{-1}b^{-1}[y_2y_5] & \\ &\sim w_0 ca^{-1}c^{-1}a[y_1y_4y_3y_2y_5] & \text{Figure 3.4.2c} \\ &\sim w_0 aba^{-1}b^{-1}[y_1y_4y_3y_2y_5] & \text{relabelling.} \end{aligned}$$

□

Now, the only thing left to show is that a connected sum of projective planes and tori is equivalent to a connected sum of projective planes.

Lemma 3.4.5. Any proper scheme w of the form $w_0(cc)(aba^{-1}b^{-1})w_1$ is equivalent to the scheme $w' = w_0(aabbcc)w_1$.

Proof. Consider the sequence of operations:

$$\begin{aligned} w_0(cc)(aba^{-1}b^{-1})w_1 &\sim cc[ab][a^{-1}b^{-1}][w_1w_0] & \text{permuting} \\ &\sim cc[ab][ba]^{-1}[w_1w_0] & \text{inverse substitution} \\ &\sim [ab]c[ba]c[w_1w_0] & \text{Lemma 3.4.2} \\ &= [a]b[c]b[acw_1w_0] & \\ &\sim bb[ac^{-1}acw_1w_0] & \text{Lemma 3.4.2} \\ &= [bb]a[c]^{-1}a[cw_1w_0] & \\ &\sim aa[bbccw_1w_0] & \text{Lemma 3.4.2} \\ &\sim w_0aabbccw_1 & \text{permuting.} \end{aligned}$$

□

In particular the theorem above states that the space $X = T_1\#\mathbb{RP}^2$ is homeomorphic to $\mathbb{RP}_3 = \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$. We can visualize it as on Figure 3.4.3.

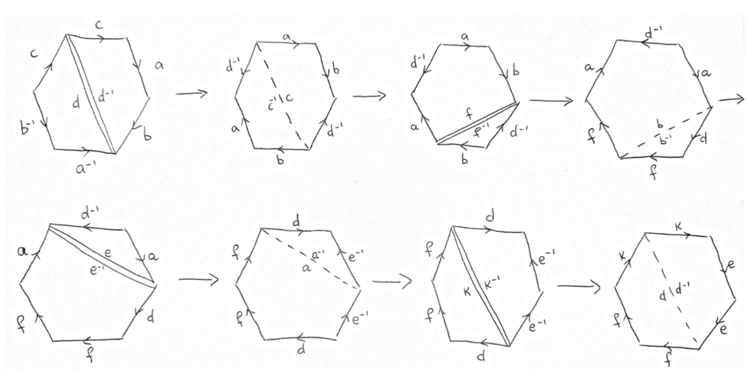


Figure 3.4.3: Elementary scheme operations showing that $T^2\#\mathbb{RP}^2 \cong \mathbb{RP}_3$.

Theorem 3.4.6 (The Classification Theorem). *Let X be the quotient space obtained from a polygonal region in the plane by pasting its edges together in pairs. Then X is homeomorphic either to S^2, to the n-fold torus T_n, or to the m-fold projective plane $\mathbb{RP}_m$.*

Proof. Let w be a proper labelling scheme of length at least 4 for the quotient space X from the polygonal region P. We would like to show that w is equivalent to one of the schemes:

1. $aa^{-1}bb^{-1}$, which produces a 2-sphere;
2. $(a_1b_1a_1^{-1}b_1^{-1})(a_2b_2a_2^{-1}b_2^{-1})\dots(a_nb_na_n^{-1}b_n^{-1})$ with $n \geqslant 1$, which produces T_n;
3. $abab$, which produces $\mathbb{RP}_1$;
4. $(a_1a_1)(a_2a_2)\dots(a_ma_m)$ with $m \geqslant 2$, which produces $\mathbb{RP}_m$.

Let's consider w to be a proper scheme of torus type. Using the method of mathematical induction, we will show that w is equivalent either to the scheme (1) or to the scheme (2). If w has length 4, then it has to be either $aa^{-1}bb^{-1}$ or $aba^{-1}b^{-1}$, where the first one is a scheme of type (1) and the second one is of type (3). Assume w has length greater than 4. If w is equivalent to a shorter scheme of torus type, then by induction hypothesis, it is equivalent either to the scheme (1) or to the scheme (2). Otherwise, w cannot contain two adjacent elements having the same label. By Lemma 3.4.4 (taken with empty w_0), w is equivalent to a scheme having the same length as w but of the form $aba^{-1}b^{-1}w_3$, where w_3 is a non-empty scheme of torus type. Similarly to w, w_3 cannot contain any two adjacent terms having the same label. Thus, we can use Lemma 3.4.4 again with $w_0 = aba^{-1}b^{-1}$ and, as a result, w has to be equivalent to the scheme of the form

$$(aba^{-1}b^{-1})(cdc^{-1}d^{-1})w_4,$$

where w_4 is a either empty or of torus type. If it is empty, we are finished and the scheme is of type (2). Otherwise, we can apply the lemma again until we reach an empty scheme.
Now, let's consider w to be a proper scheme of projective type. Similarly, we are going to use induction to show that w is equivalent either to the scheme (3) or to the scheme (4). If w has length 4, it must be either $aabb$, which is a scheme of type (4), or $aab^{-1}b$, which by Lemma 3.4.2 is equivalent to the scheme $abab$, which is a scheme of type (3). Now, assume the length of w is greater than 4. By Corollary 3.4.3, w has to be equivalent to the scheme of the form

$$w' = (a_1a_1)\dots(a_ka_k)w_1,$$

where $k \geqslant 1$ and w_1 being either empty or of torus type. In case it is empty, we are done. Otherwise, if w_1 has two adjacent terms with the same label, then w' is equivalent to a shorter scheme of projective type and we can apply the induction hypothesis. Otherwise, by Lemma 3.4.4, w' is known to be equivalent to a scheme of the form

$$w'' = (a_1a_1)\dots(a_ka_k)aba^{-1}b^{-1}w_2,$$

where w_2 is either empty or of torus type. By Lemma 3.4.5, w'' is equivalent to the scheme $(a_1a_1)\dots(a_ka_k)aabbw_2$, which is a scheme of type (4). Continuing the same process, we reach the empty scheme since w is finite, and so it has to be equivalent to the scheme of type (4). □

Example 3.4.7. Consider X to be a quotient space obtained from an 8-sided polygonal region P by means of the labelling scheme $abcdad^{-1}cb^{-1}$. See Figure 3.4.4. Let $\pi : P \to X$ be the quotient map. We can not use Theorem 3.1.8 since we do not have all vertices mapped to a single point. Instead, we end up with 2 points x_0 and x_1.
In this case we can still calculate the fundamental group of the boundary of X. We can see that a connects x_1 to itself, c connects x_0 to itself, while b and d are paths between x_0 and x_1 of opposite direction. We now want to calculate its fundamental group. Note that we can retract the segment d into the point x_0, making the point coincide with x_1. The resulting deformation retract ends up being the wedge of three circles. Thus, $\pi_1(A, x_0) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ by Theorem 2.4.2. One can also show it using labelling schemes, where [ ] denotes an empty scheme, with the following operations.

$$
\begin{aligned}
[\,]a[bcd]a[d^{-1}cb^{-1}] &\sim aad^{-1}c^{-1}b^{-1}d^{-1}cb^{-1} && \text{by Lemma 3.4.2}\\
\sim [aa]d^{-1}[c^{-1}b^{-1}]d^{-1}[cb^{-1}] &\sim d^{-1}d^{-1}aabccb^{-1} && \text{by Lemma 3.4.2}\\
&\sim b^{-1}cd^{-1}d^{-1}aabc && \text{permuting}\\
\sim [b^{-1}]c[d^{-1}d^{-1}aab]c[\,] &\sim ccb^{-1}b^{-1}a^{-1}a^{-1}d^{-1}d^{-1} && \text{Lemma 3.4.2}\\
&\sim aabbccdd && \text{relabelling.}
\end{aligned}
$$

Therefore, X is homeomorphic to $\mathbb{RP}_3$.

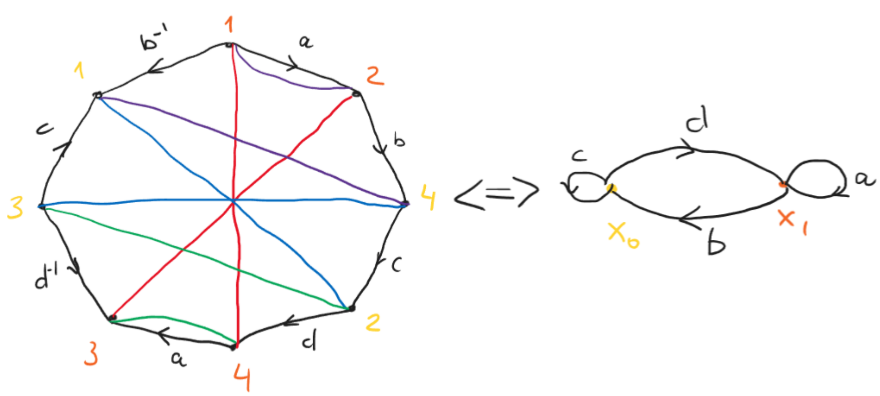


Figure 3.4.4: Polygonal region in Example 3.4.7.

3.5 Construction of Compact Surfaces

So far we have proved that every compact connected surface is homeomorphic to a surface from the list in the Theorem 3.4.6, but we have not clearly showed that every surface can be obtained by pasting together in pairs the edges of a polygonal region.

Definition 3.5.1. Let X be a compact Hausdorff space. A **curved triangle** in X is a subspace A of X and a homeomorphism $h : T \to A$, where T is a closed triangular region in the plane. If e is an edge of T, we say that $h(e)$ is an edge of A. Similarly, $h(v)$ is a vertex of A if v is a vertex of T. A **triangulation** of X is a collection of curved triangles $\{A_i\}_{i=1}^n$ in X such that $\bigcup_{i=1}^n A_i = X$ and for $i \neq j$ the intersection $A_i \cap A_j$ is either empty or a vertex or edge of A_i and A_j. If X has a triangulation, we say that X is **triangulable**.

Note that if $h_i : T_i \to A_i$ is the associated homeomorphism, then if $A_i \cap A_j$ is an edge e of both A_i and A_j, then the map $h_j^{-1}h_i$ defines a linear homeomorphism $h_j^{-1}h_i|_e$ of the edge $h_i^{-1}(e)$ of T_i with the edge $h_j^{-1}(e)$ of T_j.

Theorem 3.5.2. *Every compact surface is triangulable.*

The proof of the theorem above is a well-known result of topology. It uses Jordan curves and the interested reader can find it in [Tho92] or in [AS60]. Prior to proving the main result, we outline a few propositions to make the main proof easier.

Proposition 3.5.3. If X is a triangular region in the plane and if x is an interior point of one of the edges of X, then x does not have a neighborhood in X homeomorphic to an open 2-ball.

Proof. Suppose there is a neighbourhood U of x which is homeomorphic to an open ball B in $\mathbb{R}^2$ with the homeomorphism carrying x to 0. Note that the space $X\backslash\{x\}$ is homeomorphic to a circle. Let V be an open neighbourhood of 0 contained in B. Choose ϵ such that the open ball B_ϵ of radius ϵ centered at 0 lies in V. Consider the inclusion mappings:

$$\begin{tikzcd} B_\epsilon\backslash\{0\} \arrow[rr, "i"] \arrow[dr, "j"'] & & B\backslash\{0\} \\ & V-0 \arrow[ur, "k"'] & \end{tikzcd}$$

The inclusion i is homotopic to the homeomorphism $h(x) = x/\epsilon$, which is scaling the circle, so by Theorem 1.4.6 it induces an isomorphism of fundamental groups. Therefore, k_* must be surjective and so $V\backslash\{0\}$ cannot be simply-connected. However, a point x on the edge, which is a part of the boundary of the trianglular region, has arbitrary small neighbourhood W for which $W\backslash\{x\}$ is simply-connected. Thus, we reached a contradiction and, as a result, x does not have a neighborhood in X homeomorphic to an open 2-ball. $\square$

Proposition 3.5.4. Let X be the union of k triangles in $\mathbb{R}^3$, each pair of which intersect in the common edge e. If $k \geqslant 3$, then a point x of e does not have a neighborhood in X homeomorphic to an open 2-ball.

Proof. We would like to show that there is no neighbourhood W of x in X such that $W \backslash \{x\}$ has abelian fundamental group (since an open 2-ball without an interior point is homotopic to a circle and has fundamental group $\mathbb{Z}$). Consider a union A of all the edges of triangles of X which are different from e. The space A is a collection of k "arcs", each pair of which intersects in their endpoints. If B is the union of three of the arcs that make up A, then there is a retraction r of A onto B, obtained by mapping each of the arcs not in B homeomorphically onto one of the arcs in B, keeping the end points fixed. Then r_* is an epimorphism by Lemma 1.4.2. Since the fundamental group of B is not abelian, neither is the fundamental group of A by Theorem 1.4.6. It follows that the fundamental group of $X \backslash \{x\}$ is not abelian since A is a deformation retract of $X \backslash \{x\}$.

Assume x is the origin in $\mathbb{R}^3$. If W is an arbitrary neighbourhood of 0, we can find a scaling map $f(x) = \epsilon x$ which carries X into W. The image $X_\epsilon = f(X)$ is a copy of X lying inside of W. Consider the inclusion mappings:

$$\begin{tikzcd}
X_\epsilon \backslash \{0\} \arrow[rr, "i"] \arrow[dr, "j"'] & & X \backslash \{0\} \\
& W - 0 \arrow[ur, "k"'] &
\end{tikzcd}$$

Similarly to the proof of Proposition 3.5.3, k_* is surjective, and so the fundamental group of $W \backslash \{0\}$ cannot be abelian. □

Theorem 3.5.5. *If X is a compact surface, then X is homeomorphic to the quotient space obtained from a collection of disjoint triangular regions in the plane by pasting their edges together in pairs.*

Proof. Since the surface is compact, by Theorem 3.5.2, X is triangulable. Let $A_1, A_2, \dots, A_n$ be a triangulation of X with corresponding homeomorphisms $h_i : T_i \to A_i$. One can get any triangulation to be disjoint, thus, consider the case when the triangles T_i are already disjoint. Then the maps h_i can be combined to form a map $h : T_1 \cup T_2 \cup \ldots \cup T_n \to X$. Note that this map is a quotient map since the space $E = T_1 \cup T_2 \cup \ldots \cup T_n$ is compact and X is Hausdorff. Moreover, because the map $h_j^{-1} \circ h_i$ is linear when $A_i \cap A_j$ is an edge, h pastes the edges of T_i and T_j together by a linear homeomorphism.

First, we need to show that for each edge e, which belongs to the triangulation triangle A_i, there is exactly one other triangle A_j such that $A_i \cap A_j = e$. Note that by Proposition 3.5.3, there is at least one additional triangle A_j having e as an edge and by Proposition 3.5.4 there is only one such triangle. Therefore, the quotient map actually pastes the edges of triangles together in pairs, since each edge appears exactly twice in a scheme.

Now, we would like to show that if the intersection $A_i \cap A_j$ equals a vertex v, then there is a sequence, as visualized in Figure 3.5.0, starting with A_i and ending with A_j, of triangles having v as a vertex such that the intersection of each triangle of the sequence with its successor is an edge of each. In other words, we cannot have a case of the wedge of multiple surfaces. To show that such situation is not possible, given a common vertex v, define two triangles A_i and A_j such that $v \in A_i \cap A_j$ to be *equivalent* if there exists a sequence of triangles as mentioned above.

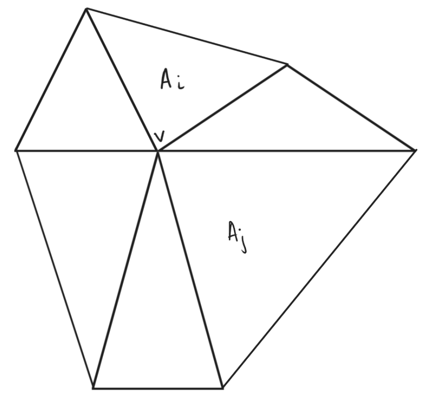


Figure 3.5.0: Visualization of the triangle sequence with a common vertex v.

Suppose there are two equivalence classes of triangles, and let B and C be the unions of the triangles in two different equivalence classes. Intersection of the sets B and C consists of v alone since no triangle in B that has a common edge with a triangle in C. Therefore, for every sufficiently small neighbourhood W of v in X, the space $W \setminus \{v\}$ is disconnected, which contradicts the locally Euclidean property of the surface X. □

Theorem 3.5.6. *If X is a compact connected triangulable surface, then X is homeomorphic to the quotient space obtained from a polygonal region in the plane by pasting their edges together in pairs.*

Proof. From the previous theorem there is a collection $\{T_i\}_{i=1}^n$ of disjoint triangular regions in the plane such that X is homeomorphic to the quotient space obtained from the collection by pasting their edges together in pairs. To extend the previous theorem, we paste the edges of triangles with the same label together. If two triangular regions have edges with the same label, we can paste the regions together along these two edges. The result would be one four-sided region with still proper orientations and labels instead of two triangular regions. Continue similarly as long as there are two regions having edges bearing the same label. Eventually, one reaches the situation with either a region with all different labels (exactly what we need) or with multiple polygonal regions, no two of which have edges bearing the same label. In such case the space ends up not being connected, which is not possible by the assumption. □

Appendices

Appendix A

Topology

The following chapter is using materials from the textbooks *Topology*[Mun00] and *General Topology*[Wil04]. As any other axiomatic branch of mathematics, we would like to start with a set of definitions and axioms, which later would develop theorems and propositions.

Definition A.0.1. A **topology** on a non-empty set X is a collection $\mathcal{T}$ of subsets of X, called **open sets**, satisfying

- Both X and $\varnothing$ are open, i.e., $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$.
- The union of any family of open subsets is open.
- The intersection of any finite family of open subsets is open.

A pair $(X, \mathcal{T})$ consisting of a set X together with a topology $\mathcal{T}$ on X is called a **topological space**.

Example A.0.2. Let $X = \{a, b, c, d, e, f\}$ and $\mathcal{T}_1 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Then $\mathcal{T}_1$ is a topology on X as it satisfies all the conditions from the definitions. On the other hand, the collection $\mathcal{T}_2 = \{X, \varnothing, \{a\}, \{a, c, d\}, \{b, c, d, e, f\}\}$ is not a topology on X since $\{b, c, d, e, f\} \cap \{a, c, d\} = \{c, d\}$, which does not belong to $\mathcal{T}_2$.

Example A.0.3. Define $\mathcal{T}$ as a collection of all subsets of X. It clearly satisfies all the conditions for a topology. We call this topology **discrete** on the set X. In this case, we call $(X, \mathcal{T})$ a **discrete space**. We can also define the topology with the smallest number of elements $\mathcal{T} = \{\varnothing, X\}$ for a set X. This topology is called **indiscrete** and the pair $(X, \mathcal{T})$ is called **indiscrete space**.

Definition A.0.4. Let $(X, \mathcal{T})$ be a topological space. A subset S of X is said to be a **closed set** in $(X, \mathcal{T})$ if its complement $X \backslash S$ is open in $(X, \mathcal{T})$.

One can define a topology based on closed sets instead of open sets. Then the words "intersection" and "union" in the definition flip: we would have the intersection of any number

of closed sets is a closed set together with the union of any finite number of closed sets being a closed set.
Note that despite the names, open and closed sets are not mutually exclusive. For instance, in discrete space every set is both open and closed - we call such sets **clopen** - while in an indiscrete space $(X, \mathcal{T})$ all subsets of X except X and $\varnothing$ are neither open or closed.

Definition A.0.5. A collection $\mathcal{B}$ of open subsets of X is a **basis** for the topology of X if every open subset of X is the union of some collection of elements of $\mathcal{B}$.

In other words, if $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on a set X, then a subset U of X is in $\mathcal{T}$ if and only if it is a union of elements of $\mathcal{B}$.

Definition A.0.6. Let Y be a non-empty subset of a topological space $(X, \mathcal{T})$. The **induced topology** on Y or the **subspace topology** is defined as $\mathcal{T}|_Y = \{U \cap Y | U \subset \mathcal{T}\}$.

Example A.0.7. Let $X = \{a, b, c, d, e, f\}$, $Y = \{b, c, e\}$ and define $\mathcal{T}$ as in Example A.0.2, i.e., $\mathcal{T} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$. Then the subspace topology on Y is

$$\mathcal{T}_Y = \{Y, \varnothing, \{c\}\}$$

Recall from the set theory the notion of equivalence relations.

Definition A.0.8. A relation $\sim$ on a set X is said to be an **equivalence relation** if it is refrexive ($x \sim x$), symmetric (if $x \sim y$ then $y \sim x$) and transitive (if $x \sim y$ and $y \sim z$ then $x \sim z$). For an element $x \in X$ the **equivalence class** is defined as all elements that are related to x:

$$[x] := \{y \in X | x \sim y\}.$$

The set of all equivalence classes of X determines a **partition** of X.

Definition A.0.9. A map $f : X \to Y$ between topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is **continuous** if $f^{-1}(U)$ is open in X for every open set U of Y.

Definition A.0.10. Topological spaces $(X, \mathcal{T}_X$ and $(Y, \mathcal{T}_Y)$ are said to be **homeomorphic** if there exists a continuous function $f : X \to Y$ such that f is bijective and has a continuous inverse. The map f is said to be a **homeomorphism** between $(X, \mathcal{T}_X$ and $(Y, \mathcal{T}_Y)$. We would write $(X, \mathcal{T}_X) \cong (Y, \mathcal{T}_Y)$. One can show that $\cong$ is an equivalence relation.

A continuous map $f : X \to Y$ is said to be a **local homeomorphism** if every point $p \in X$ has a neighbourhood $U \subseteq X$ such that $f(U)$ is open in Y and f restricts to a homeomorphism from U to $f(U)$.

Definition A.0.11. Let A be a subset of a topological space $(X, \mathcal{T})$. A point $x \in X$ is said to be a **limit point** of A if every open set U containing x also contains a point of A different from x.

Example A.0.12. Consider the topological space $(X, \mathcal{T})$, where $X = \{a, b, c, d, e\}$ and $\mathcal{T} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$. Consider $A = \{a, b, c\}$. Then elements b, d and e are limit points of A, while a and c are not.

Proposition A.0.13. Let A be a subset of a topological space $(X, \mathcal{T})$. Then A is closed in $(X, \mathcal{T})$ if and only if A contains all of its limit points.

Definition A.0.14. Let A be a subset of a topological space $(X, \mathcal{T})$. Then the set $A \cup A'$ consisting of A and all its limit points, denoted as a set A', is called the **closure** of A and is denoted by $\overline{A}$.

Definition A.0.15. Let $(X_1, \mathcal{T}_1), \ldots, (X_k, \mathcal{T}_k)$ be topological spaces. The collection of all subsets of $X_1 \times \ldots \times X_k$ of the form $U_1, \ldots, U_k$, where each U_j is open in X_j, forms a basis for a **product topology** on $X_1 \times \ldots \times X_k$.

Definition A.0.16. If $\pi : X \to Y$ is a map, a subset $U \subseteq X$ is said to be **saturated with respect to** π if U is the entire preimage of its image: $U = \pi^{-1}(\pi(U))$.

Definition A.0.17. Let X be a topological space, Y be a set and $\pi : X \to Y$ be a surjective map. The **quotient topology on** Y **determined by** π is defined by the following rule: $U \subseteq Y$ is open if and only if $\pi^{-1}(U)$ is open in X. If Y is a topological space itself, the map π is called the **quotient map** if it is surjective and continuous and Y has a quotient topology determined by π.

Here are some useful properties of a quotient map $\pi : X \to Y$:

- If B is a topological space, a map $F : Y \to B$ is continuous if and only if $F \circ \pi : X \to B$ is continuous.
- The quotient topology is the unique topology on Y for which the previous property holds.
- A subset $K \subseteq Y$ is closed if and only if $\pi^{-1}(K)$ is closed in X.
- If π is injective, then it is a homeomorphism.
- If $U \subseteq X$ is a saturated open or closed subset, then the restriction $\pi|_U : U \to \pi(U)$ is a quotient map.
- Any composition of π with another quotient map is again a quotient map.

Theorem A.0.18. *Let X and Y be topological spaces and let $F : X \to Y$ be a continuous map that is either open or closed.*

1. *If F is surjective, then it is a quotient map.*
2. *If F is injective, then it is a topological embedding.*
3. *If F is bijective, then it is a homeomorphism.*

Definition A.0.19. Let X be a topological space, $\sim$ be an equivalence relation on X, $X/\sim$ be the set of all equivalence classes of X and $\pi : X \to X/\sim$ be a natural projection sending each element $x \in X$ to its equivalence class $[x]$. Endowed with the quotient topology determined by π, the space $X/\sim$ is called **quotient space of** X **determined by** π.

Definition A.0.20. A topological space $(X, \mathcal{T})$ is said to be **connected** if the only clopen subsets of X are X and $\varnothing$. From this, it follows that a topological space $(X, \mathcal{T})$ is not connected or **disconnected** if and only if there are non-empty open sets A and B such that $A \cap B = \varnothing$ and $A \cup B = X$.

Theorem A.0.21. *The union of a collection of connected subspaces of X with a point in common is connected.*

Definition A.0.22. A topological space X is **compact** if for every collection C of open sets of X such that $\bigcup_{A \in C} A = X$ there is a finite subcollection $F \subseteq C$ such that $\bigcup_{A \in F} A = X$.

Theorem A.0.23. *The image of a compact space under a continuous map is compact.*

Definition A.0.24. A topological space X is **Hausdorff** if $\forall\, p, q \in X$ such that $p \neq q$ there exists a pair of disjoint open subsets U and V in $\mathcal{T}_X$ such that $p \in U$, $q \in V$ and $U \cap V = \varnothing$.

Note that by this definition singleton sets of a Hausdorff space are closed. In other words, is X is a Hausdorff space with a point $x \in X$, then $X \backslash \{x\}$ is open in X. To see that, consider a point a distinct from x, By definition A.0.24, there is an open set U_a containing a but not containing x. Then $\bigcup_{a \in X \backslash x} U_a$ is open as union of open sets, but it also equal to $X \backslash \{x\}$.

Definition A.0.25. A topological space X is **second-countable** if there is a countable basis for its topology.

Definition A.0.26. Suppose that one-point sets are closed in X. Then X is said to be **normal** if for each pair (A, B) of disjoint closed sets of X, there exists disjoint open sets containing A and B, respectively. In other words, every two disjoint closed sets of X have disjoint open neighborhoods.

Note that a normal space is always Hausdorff, but only compact Hausdorff space is normal.

Lemma A.0.27. Let $\pi : E \to X$ be a closed quotient map. If E is normal, so is X.

Theorem A.0.28 (The Pasting Lemma)**.** *Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.*

Theorem A.0.29 (Extreme Value Theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous function, where X is a compact set. Then f is bounded and there exists $p, q \in X$ such that $f(p) = sup_{x \in X} f(x)$ and $f(q) = inf_{x \in X} f(x)$.*

Lemma A.0.30 (Lebesgue Number Lemma)**.** For any open cover $\mathcal{A}$ of a compact metric space X, there exists a real number $\delta > 0$, also called a **Lebesgue number** for $\mathcal{A}$, such that every open ball in X of radius δ is contained in some element of $\mathcal{A}$.

Appendix B

Category Theory

Definition B.0.1. A **category** C consists of a class $\mathsf{Ob}(\mathsf{C})$, whose elements are to be called **objects**, and a class $\mathsf{Mor}(\mathsf{C})$, whose elements are to be called **morphisms**, satisfying the following:

- For each morphism f, there are objects A and B, called the **source** and **target** of f. In this case we write $f : A \to B$.
- Given any two morphisms $f : A \to B$ and $g : B \to C$, there exists a morphism $g \circ f : A \to C$, called the **composition** of f and g.
- Given any objcet A, there is an **identity morphism** $\mathbb{1}_A : A \to A$ such that for any $f : A \to B$, $f \circ \mathbb{1}_A = f = \mathbb{1}_B \circ f$.
- Morphism composition is associative: given any two morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$, $(f \circ g) \circ h = f \circ (g \circ h)$.

Definition B.0.2. In any category C, a morphism $f : A \to B$ is called an **isomorphism** if there is a morphism $g : B \to A$ such that $f \circ g = \mathbb{1}_B$ and $g \circ f = \mathbb{1}_A$. In this case, f and g are called **inverses**, g is denoted f^{-1}, and we say that A is **isomorphic** to B.

Example B.0.3. Let's look at some important examples of categories: First, let us define $\mathsf{0}$ as an empty category (with no objects and no morphisms) and $\mathsf{1}$ as a category with one object and the identity morphism.

- Sets is the category of sets and functions between them.
- $\mathsf{Sets}_{\mathsf{fin}}$ is the category of finite sets and functions between them.
- Groups is the category of groups and group homomorphisms.
- Ab is the category of abelian groups and group homomorphisms.
- Graphs is the category of graphs and graph homomorphisms.

- $\mathsf{Vect_F}$ is the category of vector spaces over a field F and linear transformations in F.
- Top is the category of topological spaces and continuous mappings.
- Poset is the category of elements of the set and orderings.

Finally, an individual group is itself a category with exactly one object, where all the morphisms are isomorphisms. For a given group G, this category is called BG.

There are also examples of categories whose objects are sets with distinguished base points, in addition to possibly other structure.

Definition B.0.4. A **pointed set** is an ordered pair (X, p) where X is a set and p is an element of X. Similarly, one can defined objects as **pointed topological spaces** and so on. Moreover, if (X, p) and (X', p') are both pointed sets, a map $F : X \to X'$ is called a **pointed map** if $F(p) = p'$. In this case, we write $F : (X, p) \to (X', p')$.

Example B.0.5. With the definition above, let's look at categories of pointed objects:

- Set_* is a category of pointed sets and pointed maps.
- Top_* is the category of pointed topological spaces and pointed continuous maps.

Definition B.0.6. A **subcategory** of a category C is a subclass $\mathsf{Ob(D)}\subseteq \mathsf{Ob(C)}$ and a subclass $\mathsf{Mor(D)} \subseteq \mathsf{Mor(C)}$ such that any morphism in $\mathsf{Mor(D)}$ is between two objects in $\mathsf{Ob(D)}$.

For example, 0 is a subcategory of any category.

Definition B.0.7 (Types of morphisms)**.** A morphism $f : A \to B$ is called a **monomorphism** if it is left cancellative, i.e., $f \circ g = f \circ h \Rightarrow g = h$. In this case, we say that f is monic.
A morphism $f : A \to B$ is called an **epimorphism** if it is right cancellative, i.e. $g \circ f = h \circ f \Rightarrow g = h$. In this case, we say that f is epic.
A morphism is called a **bimorphism** if it is both epic and monic.
A morphism is called a **retraction** if it has a left-inverse and a **section** if it has a right-inverse.
Note that a morphism which is both a retraction and a section is an isomorphism.

Definition B.0.8. An **endomorphism** is a morphism $f : A \to A$ from an object to itself. If an endomorphism is also an isomorphism, then it is called an **automorphism**. The class of endomorphisms of an object A is denoted $\mathsf{End}(A)$ and the class of automorphisms is denoted $\mathsf{Aut}(A)$.

Definition B.0.9. A **covariant functor** (or just a **functor**) $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ between categories C and D is a mapping $\mathsf{Ob\ C} \to \mathsf{Ob\ D}$ and $\mathsf{Mor\ C} \to \mathsf{Mor\ D}$ such that:

- $\mathcal{F}$ assigns to each object $X \in \mathsf{Ob\ C}$ an object $\mathcal{F}(X) \in \mathsf{Ob\ D}$.
- $\mathcal{F}$ assigns to each morphism $f \in \mathsf{Mor_C}(X, Y)$ a morphism $\mathcal{F}(f) \in \mathsf{Mor_C}(\mathcal{F}(X), \mathcal{F}(Y))$.

- $\mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}$.
- $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

In short, a functor is a morphism of categories. In particular, every category C has the identity functor $\mathbb{1}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$.

Definition B.0.10. Two categories C and D are **isomorphic** if there exist functors $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} : \mathsf{D} \to \mathsf{C}$ that are inverses $\mathcal{F} \circ \mathcal{G} = \mathbb{1}_{\mathsf{D}}$ and $\mathcal{G} \circ \mathcal{F} = \mathbb{1}_{\mathsf{C}}$. In particular, a functor is an isomorphism functor if and only if it is bijective on the class of objects and the class of morphisms.

Definition B.0.11. Let C be a category and $\{A_i | i \in I\}$ be a family of objects of C. A **product** for the family $\{A_i | i \in I\}$ is an object P of C together with a family of morphisms $\{\pi_i : P \to A_i | i \in I\}$ such that for any object B and a family of morphisms $\{\phi_i : B \to A_i | i \in I\}$, there is a unique morphism $\phi : B \to P$ such that $\pi_i \circ \phi = \phi_i$ for all $i \in I$.

Definition B.0.12. A **coproduct** for the family $\{A_i | i \in I\}$ is an object S of C together with a family of morphisms $\{\iota_i : A_i \to S | i \in I\}$ such that for any object B and a family of morphisms $\{\psi_i : A_i \to B | i \in I\}$, there is a unique morphism $\psi : S \to B$ such that $\psi \circ \iota_i = \psi_i$ for all $i \in I$.

Theorem B.0.13. *If $(P, \{\pi_i\})$ and $(Q, \{\psi_i\})$ are both products or both coproducts of the family $\{A_i | i \in I\}$ of objects of a category C, the P and Q are equivalent.*

In many categories, the "objects" are sets or are sets with an added structure (such as groups). When this is the case, the morphisms can be considered as functions on sets.

Definition B.0.14. A **concrete** category is a category C together with a function σ that assigns to each object A of C a set $\sigma(A)$, called the *underlying set* of A, such that

1. every morphism mapping $A \to B$ of category C is a function on the underlying sets $\sigma(A) \to \sigma(B)$,
2. the identity morphism of each object A of C is the identity function on the underlying set $\sigma(A)$, and
3. composition of morphisms in C agrees with composition of functions on the underlying sets.

Definition B.0.15. Let F be an object in a concrete category C, X a nonempty set, and $i : X \to F$ a set map. Then object F is **free** on the set X provided that for any object A of C and set map $f : X \to A$, there exists a unique morphism of C, $\overline{f} : F \to A$, such that $\overline{f} \circ i = f$ as a set map $X \to A$.

Theorem B.0.16. *If C is a concrete category, F and F_0 are objects of C such that F is free on the set X and F_0 is free on the set X_0 and $|X| = |X_0|$, then F is equivalent to F_0.*

Bibliography

[AS60] L.V. Ahlfors and L. Sario. *Riemann Surfaces*. Princeton mathematical series. Princeton University Press, 1960. ISBN: 9780691626123. URL: `https://books.google.ca/books?id=4C4PAAAAIAAJ`.

[D L97] Johnson D. L. *Presentations of Groups*. Vol. 2nd ed. London Mathematical Society Student Texts Vol. 15. Cambridge University Press, 1997. ISBN: 9780521585422. URL: `https://ezproxy.tru.ca/login?url=https://search.ebscohost.com/login.aspx?direct=true&db=nlebk&AN=570482&site=eds-live&scope=site`.

[Dav87] Gale David. "The Classification of 1-Manifolds: A Take-Home Exam". In: *The American Mathematical Monthly* 94.2 (1987), pp. 170–175. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/2322421` (visited on 11/08/2022).

[Hat02] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002. ISBN: 9780521795401. URL: `https://books.google.ca/books?id=BjKs86kosqgC`.

[Hun12] T.W. Hungerford. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2012. ISBN: 9781461261018. URL: `https://books.google.ca/books?id=e-YlBQAAQBAJ`.

[Mas91] W.S. Massey. *A Basic Course in Algebraic Topology*. Graduate Texts in Mathematics. Springer New York, 1991. ISBN: 9780387974309. URL: `https://books.google.ca/books?id=svbU9nxi2xQC`.

[Mun00] J.R. Munkres. *Topology*. Topology. Prentice-Hall, 2000. ISBN: 9780131784499. URL: `https://books.google.ca/books?id=NnCjQgAACAAJ`.

[Tho92] Carsten Thomassen. "The Jordan-Schonflies Theorem and the Classification of Surface". In: *The American Mathematical Monthly* 99.2 (1992), pp. 116–131. ISSN: 00029890, 19300972. URL: `http://www.jstor.org/stable/2324180` (visited on 12/21/2022).

[Wil04] S. Willard. *General Topology*. Addison-Wesley series in mathematics. Dover Publications, 2004. ISBN: 9780486434797. URL: `https://books.google.ca/books?id=-o8xJQ7Ag2cC`.